TOPICS IN LINEAR AND NONLINEAR DISCRETE OPTIMIZATION

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TOPICS IN LINEAR AND NONLINEAR DISCRETE OPTIMIZATION

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SUMMARY

This work contributes to modeling, theoretical, and practical aspects of structured Mathematical Programming problems. Many real-world applications have nonlinear characteristics and can be modeled as Mixed Integer Nonlinear Programming problems (MINLP). Modern global solvers have significant difficulty handling large-scale instances of them. Several convexification and underestimation techniques were proposed in the last decade as a part of the solution process, and we join this trend. The thesis has three major parts.

The first part considers MINLP problems containing convex (in the sense of continuous relaxations) and posynomial terms (also called monomials), i.e. products of variables with some powers. Recently, a linear Mixed Integer Programming (MIP) approach was introduced for minimization the number of variables and transformations for convexification and underestimation of these structured problems. We provide polyhedral analysis together with separation for solving our variant of this minimization subproblem, containing binary and bounded continuous variables. Our novel mixed hyperedge method allows to outperform modern commercial MIP software, providing new families of facet-defining inequalities. As a byproduct, we introduce a new research area called mixed conflict hypergraphs. It merges mixed conflict graphs and 0-1 conflict hypergraphs.

The second part applies our mixed hyperedge method to a linear subproblem of the same purpose for another class of structured MINLP problems. They contain signomial terms, i.e. posynomial terms of both positive and negative signs. We obtain new facet-defining inequalities in addition to those families from the first part.

The last part is dedicated to managing guest flow in Georgia Aquarium after the Dolphin Tales opening with applying a large-scale MINLP. We consider arrival and departure processes related to scheduled shows and develop three stochastic models for them. If demand for the shows is high, all processes become interconnected and require a generalized model. We provide and solve a Signomial Programming problem with mixed variables for minimization resources to prevent and control congestions.

CHAPTER I

INTRODUCTION

In this chapter, we introduce some background material and give a brief overview of major tools in the thesis.

The Mathematical Programming problem is the problem of maximizing or minimizing an objective function subject to constraints and extra restrictions on a subset of the variables, if any:

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & g_k(x) \leq 0 \quad k = 1...m \\ & x \in X \end{aligned}$$

or in a shorter variant $max\{f(x) : x \in X\}$, where $f : R^n \to R$ and $X \subseteq R^n$.

Special attention is paid to the linear programming (LP) problems, where all components are linear, and the linear mixed integer programming (MIP) problems with integrality restrictions on a subset of the variables: $max\{cx + dy : Gx + Hy \leq b, x \in Z^n_+, y \in R^n_+\}$, where Z^n_+ is the set of nonnegative integer n-dimensional vectors, R^n_+ is the set of nonnegative real m-dimensional vectors, and all vectors and matrices c, d, G, H, b are rational and have respective dimensions.

The set $S = \{Gx + Hy \leq b, x \in Z^n_+, y \in R^n_+\}$ is called the feasible region, or the set of feasible solutions. We say a point (x, y) is feasible if it satisfies all constraints, i.e. $(x, y) \in S$. If no such point exists, we say that the problem is infeasible.

MIP has special cases like pure integer programming (IP), binary IP, and mixed binary IP ($x \in B^n$). We are going to use abbreviation "MIP" even for its special cases with no loss of generality.

A problem $max\{\tilde{f}(x) : x \in T \subseteq R^n\}$ is a relaxation of $max\{f(x) : x \in X \subseteq R^n\}$ if $X \subseteq T$ (i.e. has an enlarged set of feasible solution), and $\tilde{f}(x) \geq f(x)$ (i.e. has the same or better objective value everywhere). For example, the LP relaxation of MIP is obtained by dropping the integrality restrictions. The same way, we can get the continuous relaxation of mixed integer nonlinear programming (MINLP) problems.

Branch-and-bound method for solving MIPs is based on divide-and-conquer principle. First, we get a solution (x^*, y^*) of the LP relaxation of a MIP problem. If $x^* \in Z^n$, then the solution is optimal for the MIP as well. Otherwise, $\exists x_j^* \notin Z$, and we create two subproblems with additional constraints $x_j \leq \lfloor x_j^* \rfloor$ and $x_j \geq \lceil x_j^* \rceil$ respectively. This scheme is applied recursively to each of the subproblems. Each time such a new problem is created, we consider it to be a node in a branch-and-bound (or search) tree. For each active node, we solve an LP relaxation, and depending on the LP solution, we may not have to branch from this node in the cases of infeasibility, impossibility to yield a better integer solution than the known one (incumbent) so far, or obtaining a new feasible integer solution.

Another method to solving MIP problems is to use cutting planes. We, again, start from the LP relaxation of a MIP problem and iteratively strengthen this relaxation around an optimal solution by adding valid linear inequalities (i.e. satisfied by every feasible point of a MIP problem) to the formulation. Cutting planes (or cuts) are valid inequalities for which the solution of the LP relaxation in consideration is infeasible.

Branch-and-cut methodology combines branch-and-bound and cutting plane methods. LP relaxations are strengthened with cutting planes at nodes of the search tree. So called facet-defining inequalities play a special role as the strongest cuts. A few definitions from polyhedral theory are necessary to say more about them.

A polyhedron $P \subseteq R^n$ is the set of points that satisfy a finite number of linear inequalities. For a set $\tilde{S}$ of m points x^i in R^n, the convex hull of $\tilde{S}$ is defined as $conv(\tilde{S}) = \{x \in R^n : x = \sum_{i=1}^m \lambda_i x^i, \lambda_i \geq 0, i = 1...m, \sum_{i=1}^m \lambda_i = 1\}$.

A collection of k points x^i in R^n is said to be affinely independent if the unique solution of $\sum_{i=1}^k \alpha_i x^i = 0$, with $\sum_{i=1}^k \alpha_i = 0$, is $\alpha_i = 0$, $i = 1...k$.

A polyhedron $P \subseteq R^n$ is of dimension k ($dim(P) = k$), if the maximum number of affinely independent points in P is $k + 1$. If $dim(P) = n$, then P is said to be full-dimensional.

A facet-defining inequality of P is a valid inequality that is necessary to define P. Its dimension is $dim(P) - 1$.

One of the tools representing logical relations between 0-1 (or binary) variables is a conflict graph. It has a vertex for each binary variable x_i and x_j, and an edge between two vertices in the case of relations "if $x_i = 1$, then $x_j = 0$". Conflict hypergraphs generalize conflict graphs in the sense that one edge may be connecting more than two vertices.

CHAPTER II

ON CONVEXIFICATION OF MINLP CONTAINING POSYNOMIALS

2.1 Overview

In this chapter, we consider a Mixed Integer Nonlinear Programming problems (MINLP) containing posynomial terms. Recently, a linear Mixed Integer Programming (MIP) approach was introduced for minimization the number of variables and transformations in convexification and underestimation techniques for these structured problems. We provide polyhedral analysis together with separation for solving our variant of this minimization subproblem. Our novel mixed hyperedge method in conflict graphs allows to outperform modern commercial MIP software.

In recent years there has been an increasing interest in mixed integer nonlinear programming problems. Many real-word applications have nonlinear characteristics and can be modeled effectively using both discrete and continuous variables. The challenges associated with these problems, which are generally *NP-hard*, and in many cases are nonconvex in terms of continuous variables, are intriguing to optimization researchers who strive to develop advanced approaches for tackling them.

The most popular technique for the minimization problem is based on the use of convex underestimators of the nonconvex feasible region and then applying well-known techniques from linear MIP including branching, bounding, polyhedral results, and the associated cutting planes.

In some of these MINLP instances, the presence of posynomial terms (monomials), i.e. the products of variables with some powers, is common. Consider the classical Geometric Programming (GP) problem:

$$
\begin{aligned}
\min \quad & f(x) \\
\text{s.t.} \quad & g_k(x) \le 1 \quad k = 1...m \\
& x > 0 \qquad\quad \text{(and mixed integer in general)}
\end{aligned}
$$

where $f(x)$ and $g_k(x)$ are posynomial functions in the form

$$\sum_i c_i \prod_{j=1}^{n} x_j^{a_{ij}}$$

with $c_i > 0, a_{ij} \in R$. Let T be the total number of terms in $f(x)$ and all $g_k(x)$.

Boyd et al. [7] provide a recent review on this topic. In general, GP problems may have equality constraints $g_l(x) = 1$ of the same posynomial form. GP can be extended using some operations over posynomials like addition, multiplication, positive fractional power, and maximum, to generalized posynomials. Moreover, c_i can be negative (signomial programming problems) and the variables may have special domains other than just being strictly positive or having a bounded positive interval. There are papers that deal with nonpositive and free variables (e.g. Li and Tsai [27], Lin and Tsai [29], Tsai and Lin [40], Tsai et al. [41]).

In this chapter, we consider only positive terms $\prod_{j=1}^{n} x_j^{a_j}$, but assume that they can be present not only in GP problems, but also in general MINLP as parts of the objective function and constraints. Our goal is to convexify all posynomial terms of the MINLP. We assume that all other terms in the problem (constraints and/or objective function) are convex and thus do not need any transformation.

Suppose that we need to convexify I_T posynomial terms (out of T in the case of the classical GP problem above).

The following statements are well known.

Lemma 2.1.1. *The function $x^\alpha, x > 0$ is convex when $\alpha \le 0$ or $\alpha \ge 1$ and concave (i.e. $-x^\alpha$ is convex) when $0 \le \alpha \le 1$.*

Proposition 2.1.1. *(e.g. Maranas and Floudas (1997) [33] or Lundell et al. [31]) The posynomial term $\prod_{j=1}^{n} x_j^{a_j}$ is convex if*

1. $a_j \leq 0$ *for* $j = 1...n$ *or*

2. there exists one $a_p > 0$*, and all other* $a_j \leq 0, j \neq p$ *and* $\sum_{j=1}^{n} a_j \geq 1$.

And $\prod_{j=1}^{n} x_j^{a_j}$ *is concave (i.e.* $-\prod_{j=1}^{n} x_j^{a_j}$ *is convex) if* $a_j \geq 0$ *for* $j = 1, ..., n$ *and* $\sum_{j=1}^{n} a_j \leq 1$.

Researchers applied different mappings to nonconvex posynomial terms for convexification (see Floudas and Gounaris [16], and Gounaris and Floudas [18]), and the most popular of all are exponential transformations (ET) (e.g. Maranas and Floudas (1997) [34]) and power transformations (e.g. Li et al. [28], Lu et al. [30]). A power transformation can be positive (PPT) or negative (NPT), depending on the case as shown in the Proposition above. In NPT all powers in the convexified term are negative. In all approaches, there is a compromise between the computational effort and the tightness of underestimation. ET has served as a classical tool applied to continuous GP problems before the resulting problems are solved by general methods for convex problems. ET is related to positive terms only and it has computational restrictions as indicated in Ben-Tal and Nemirovski [4]. It has been shown to always give a tighter convex underestimation than NPT (Lundell and Westerlund [32]). PPT also gives a tighter convex underestimation than NPT under some additional conditions. In general, PPT and ET have a parity: neither has an advantage for the whole domain of the variables, while for univariate functions PPT performs better.

We consider power transformations $\tilde{x}_{ij}^{q_{ij}}$, where $\tilde{x}_{ij}$ corresponds to the transformation of x_j in posynomial term i. Extra conditions for powers q_{ij} have to be set. The transformed problem is still nonconvex because equality constraints corresponding to the relations between the original and transformed variables have to be included. To remove these nonconvexities, piecewise linear approximations can be applied to the inverse transformations $\tilde{x}_{ij} = x_{ij}^{1/q_{ij}}$. Westerlund and his coauthors in a series of papers (Bjork et al. [6], Lundell et al. [31], Pörn et al. (2008) [35], Pörn et al. (1999)

[36], Westerlund [44]) and references therein considered power transformations for signomial problems with consequent piecewise linear approximations and the computational framework based on the Extended Cutting Plane Method. The recent work by Lundell et al. [31] addressed the problem of the optimal selection of the variables that need to be transformed.

To make further contribution to this research area, we consider a minimization subproblem of the total number of transformations and original variables to be transformed to get the convexification of posynomial terms. Some of our constraints are similar to those in Lundell et al. [31] because they come from the same implications. We rigorously describe our modeling steps from scratch, remove big-Ms, and adjust this approach to large-scale problems by analyzing its polyhedral structure with use of the conflict graph and hypergraph theory (Lee [24], Atamturk et al. [2], Easton et al. [13], Lee and Maheshwary [26]).

The main prerequisites for conflict structures considered in this work come from mixed conflict graphs (introduced in Atamturk et al. [2] with a predecessor work by Johnson [23]) on one hand, and 0-1 conflict graphs and hypergraphs (see, for example, Lee [24], Bixby and Lee [5], Euler et al. [14], Easton et al. [13] and preferences therein). We also provide new results in hypergraph structures. The mixed hyperedge, mixed star-clique inequalities, and weighted complementary inequalities are new terms in the theory.

The outline of this chapter is as follows. In Section 2.2, we present the MIP model. Section 2.3 describes probing and conflict graph construction to support our main results in Section 2.4. Section 2.5 is dedicated to solving separation problems for derived facet-defining inequalities. Finally, computational results are presented in Section 2.6.

2.2 Convexification and the MIP Formulation.

To provide the linear MIP formulation of the convexification problem we first introduce the sets of binary variables.

Let $y_{ij} = \begin{cases} 1, & \text{if } x_j \text{ in the ith term is transformed} \\ 0, & \text{otherwise} \end{cases}$

and $t_j = \begin{cases} 1, & \text{if } x_j \text{ is transformed in any of the terms where it is found} \\ 0, & \text{otherwise} \end{cases}$

We apply power transformations on posynomial terms of the initial nonlinear problem. The transformed term i becomes $c_i \prod_{j=1}^{n} x_j^{(1-y_{ij})a_{ij}} \tilde{x}_{ij}^{y_{ij}a_{ij}q_{ij}}$.

We have $\sum_{i=1}^{I_T} y_{ij} \leq I_T t_j \quad \forall j$, or

$$y_{ij} \leq t_j \quad \forall i, j \tag{1}$$

Unlike a generic facility location case, here these two are equivalent because of having 0-1 variables y_{ij} instead of continuous ones.

The total number of transformed original variables is $\sum_{j=1}^{n} t_j$. So, we have in the objective function

$$min \ \sum_{j=1}^{n} t_j + \sum_{i=1}^{I_T} \sum_{j=1}^{n} y_{ij}$$

Let

$$s_{ij} = \begin{cases} 1, & \text{if the power of } x_j \text{ in term } i \text{ remains} > 0 \text{ after the transformation} \\ 0, & \text{otherwise} \end{cases}$$

We have

$$\sum_{j=1}^{n} s_{ij} \leq 1 \quad \forall i \tag{2}$$

For $a_{ij} \le 0$ we fix all variables: $q_{ij} = 1$, $y_{ij} = 0$, $s_{ij} = 0$. In general, depending on the existence of the power p from the Proposition, we have the following relationships between q_{ij} and a_{ij}:

$$q_{ij} \begin{cases} = 1, & \text{if } a_{ij} \le 0 \text{ (no transformation)} \\ < 0, & \text{if } a_{ij} > 0 \text{ and } j \ne p \text{ (change to negative)} \\ \ge 1, & \text{if } a_{ij} > 0 \text{ and } j = p \text{ (the only one power remains positive)} \end{cases}$$

Why we do not include the values between 0 and 1 is explained in Lundell et al. [31] and references therein. The point is that in addition to being convexified by the transformations, the posynomial terms should also be underestimated for further use. It requires the condition $c_i \prod_{j=1}^{n} \hat{x}_{ij}^{a_{ij}q_{ij}} \le c_i \prod_{j=1}^{n} \tilde{x}_{ij}^{a_{ij}q_{ij}}$ for term i, where $\hat{x}_{ij}$ are the piecewise linear approximations of the inverse transformations $\tilde{x}_{ij}$. For the variable remaining with a positive power after the transformation, $\tilde{x}_{ip} = x_{ip}^{1/q_{ip}}$ is considered as an increasing function with the condition $\hat{x}_{ip}^{a_{ip}q_{ip}} \le \tilde{x}_{ip}^{a_{ip}q_{ip}} \iff \hat{x}_{ip} \le \tilde{x}_{ip}$. It demands the concavity of $\tilde{x}_{ip}$. Thus, $q_{ip} \ge 1$. The other two cases are not affected. Indeed, the variables with negative powers do not need to be transformed ($q_{ip} = 1$) and, therefore, there is no need of the piecewise linear approximations. Besides, $\tilde{x}_{ip}$ ($j \ne p$) are decreasing functions and the condition $q_{ij} < 0$ ($j \ne p$) holds.

If p exists, then $\sum_{j=1}^{n} q_{ij}a_{ij} \ge 1$. Using classical IP formulation approaches (Williams [45]), we can write

$$\sum_{j=1}^{n} q_{ij}a_{ij} + \tilde{m}_i \sum_{j=1}^{n} s_{ij} \ge \tilde{m}_i + 1 \quad \forall i \tag{3}$$

(if p exists, $\sum_{j=1}^{n} s_{ij} = 1$), where $\tilde{m}_i$ is the lower bound of $\sum_{j=1}^{n} q_{ij}a_{ij} - 1$.

Now consider only $a_{ij} > 0$. For all such i and j, when $s_{ij} = 1$, we have $q_{ij} \ge 1$; and when $s_{ij} = 0$, we have $q_{ij} < 0$.

So,

$$q_{ij} + m_1 s_{ij} \ge m_1 + 1 \quad \forall i, j \text{ s.t. } a_{ij} > 0, \tag{4}$$

where m_1 is the lower bound of $q_{ij} - 1$, and

$$q_{ij} \leq \varepsilon(s_{ij} - 1) - m_2 s_{ij} \quad \forall i, j \ s.t. \ a_{ij} > 0, \tag{5}$$

where m_2 is the lower bound of $-q_{ij}$ and ε is a small number.

The relationships of y_{ij} with s_{ij} and q_{ij} are summarized in Table 1.

Table 1: $s_{ij}, y_{ij},$ *and* q_{ij} relations

$s \setminus q$	< 0	1	> 1
0	$y = 1$	$y = 0$ (for $a_{ij} \leq 0$ only)	$\times$
1	$\times$	$y = 0$	$y = 1$

It follows:

$$y_{ij} + s_{ij} \geq 1 \quad \forall i, j \ s.t. \ a_{ij} > 0 \tag{6}$$

$$q_{ij} - 1 \geq m_1 y_{ij} \quad \forall i, j \ s.t. \ a_{ij} > 0, \tag{7}$$

where m_1 is again the lower bound of $q_{ij} - 1$,

$$q_{ij} - 1 \leq M_1 y_{ij} \quad \forall i, j \ s.t. \ a_{ij} > 0, \tag{8}$$

where M_1 is the upper bound of $q_{ij} - 1$,

$$y_{ij} \leq (1 - \varepsilon) q_{ij} + M(1 - s_{ij}) \quad \forall i, j \ s.t. \ a_{ij} > 0, \tag{9}$$

if we accept $q_{ij} > 1$ as $q_{ij} \geq \frac{1}{1-\varepsilon}$. Thus, q_{ij} are semicontinuous variables, $q_{ij} \in [-M_i, -\varepsilon]$, $\{1\}$, $[\frac{1}{1-\varepsilon}, M_i]$, i.e. $[-M_i, 0)$, $\{1\}$, $(1, M_i]$.

de Farias considered semi-continuous knapsack polyhedron [15] and introduced "three-term semi-continuous constraints $x_j \in [0, \alpha_j] \cup \{\beta_j\} \cup [\gamma_j, \eta_j], j \in N$" like ours for q_{ij}. We are not aware of any general polyhedral study for problems with such constraints.

It is convenient to take $-u \leq q_{ij} \leq u$ for all i, j such that $a_{ij} > 0$, $u > 1$ and $\varepsilon = \frac{1}{u}$. One way how to choose u and $\varepsilon = \frac{1}{u}$ is the following. Keeping in mind the future inverse transformations, Lundell et al. [31] indicate that the values of q_{ij} closer to 1 or -1 are numerically more stable than those close to 0 or large negative values. Thus it is desirable to have the right boundary in $[-M_i, -\varepsilon]$ close to -1 and the left boundary in $[\frac{1}{1-\varepsilon}, M_i]$ close to 1, which can be found by solving the problem

$$min_{\varepsilon>0} \ \ (1-\varepsilon)^2 + (\frac{1}{1-\varepsilon} - 1)^2,$$

i.e. $\varepsilon \approx 0.2755$ and $u^* \approx 3.63$, or roughly $u^* = 4$.

The validness of the model is verifiable by the following trivial feasible solution, related to the pure negative transformation:

$$s_{ij} = 0 \ \ \forall i, j;$$

$$y_{ij} = \begin{cases} 1, & \text{if } a_{ij} > 0 \\ 0, & \text{if } a_{ij} \leq 0 \end{cases}$$

$$q_{ij} = \begin{cases} -u, & \text{if } a_{ij} > 0 \\ 1, & \text{if } a_{ij} \leq 0 \end{cases}$$

$$t_j = 1 \ \forall j, \text{or simply } t_j = max_i \ y_{ij} \ \forall j.$$

Constraints (1)-(9) can now be written with using only variables t_j, y_{ij}, s_{ij}, and given data a_{ij} and u. For the purpose of a conflict graph construction in the next section, we introduce complementary variables $\bar{t}_j \triangleq 1 - t_j$, $\bar{y}_{ij} \triangleq 1 - y_{ij}$, $\bar{s}_{ij} \triangleq 1 - s_{ij}$ and $\tilde{q}_{ij} \triangleq q_{ij} + u$ (to have $0 \leq \tilde{q}_{ij} \leq 2u$ with $\bar{\tilde{q}}_{ij} \triangleq 2u - \tilde{q}_{ij}$) and denote J_i^-, J_i^0, and J_i^+ the sets of index j in posynomial term i for $\{j : a_{ij} < 0\}$, $\{j : a_{ij} = 0\}$ and $\{j : a_{ij} > 0\}$, respectively.

Next, we refine and adjust the constraints (see Appendix A). In particular, transforming constraint 12, we assume that $u \sum_{j \in J_i^+} a_{ij} \geq 1 - \sum_{j \in J_i^-} a_{ij}$ (otherwise, power p does not exist for term i). It is also shown that small $\sum_{j \in J_i^+} a_{ij}$ may suggest choosing $u > 4$. Thus, parameter u may or may not take into account the data a_{ij}. In general, this situation introduces the trade-off between choosing a relatively small value for parameter u and admitting fixing some variables a priori (i.e. $\sum_{j \in J_i^+} s_{ij} = 0$ here). The next section brings two more assumptions, i.e. the condition for not fixing s_{ij} at 0, which is stronger than our current assumption, and the condition for not fixing y_{ij} at 1, which, in turn, stronger than not fixing s_{ij} at 0. This hierarchy of three assumptions will play an important role in the full-dimensional analysis of Section 4 and designing our preprocessor for the computations.

Besides, we add the term $-\frac{1}{2uI_T J} \sum_i \sum_j \tilde{q}_{ij}$ (i.e. a scaled averaged $\tilde{q}$-value, where $J = |J_i^+| \ \forall i$ is assumed) to the objective function in order to discourage negative transformations of big magnitudes. In other words, $\tilde{q}_{ij}$ should not be close to 0. We emphasize that this choice of the q-term in the objective function is without any loss of generality for our analysis of the polyhedral structure. The original paper with this type of optimization problem (Lundell et al. [31]) mentions that the choice can be flexible. In particular, they consider the deviations of q_{ij} from desirable values for future computations in their framework (which require extra constraints in the model) and introduce weighted coefficients for the terms in the objective function. So, the q-term comes usually from a particular goal. We provide one more form of the objective function later in the computational part of this chapter.

We now summarize the MIP formulation:

$$min \ \sum_j t_j + \sum_i \sum_j y_{ij} - \frac{1}{2uI_T J} \sum_i \sum_j \tilde{q}_{ij}$$

s.t.

(10) $y_{ij} \leq t_j \quad \forall i, j$

(11) $\sum_{j \in J_i^+} s_{ij} \leq 1 \quad \forall i$

(12) $-\sum_{j \in J_i^+} a_{ij} \tilde{q}_{ij} + (-\sum_{j \in J_i^-} a_{ij} + u \sum_{j \in J_i^+} a_{ij} + 1) \sum_{j \in J_i^+} s_{ij} \leq 0 \quad \forall i$

(13) $-\tilde{q}_{ij} + (u+1) s_{ij} \leq 0 \quad \forall i, j \in J_i^+$

(14) $\tilde{q}_{ij} - (u + \frac{1}{u}) s_{ij} \leq u - \frac{1}{u} \quad \forall i, j \in J_i^+$

(15) $y_{ij} + s_{ij} \geq 1 \quad \forall i, j \in J_i^+$

(16) $-\tilde{q}_{ij} - (u+1) y_{ij} \leq -(u+1) \quad \forall i, j \in J_i^+$

(17) $\tilde{q}_{ij} - (u-1) y_{ij} \leq u+1 \quad \forall i, j \in J_i^+$

(18) $-(u-1)\tilde{q}_{ij} + u y_{ij} + u^2 s_{ij} \leq u \quad \forall i, j \in J_i^+$

$t, y, s \in \{0, 1\}, \tilde{q} \in [0; 2u]$

Denote S the mixed 0-1 set defined by these inequalities and conv (S) its convex hull.

2.3 Probing and Conflict Graph Construction.

We next build a conflict graph related to the mixed vertex packing polytope (Atamturk et al. [2])

$MVP = \{z \in B^n, v \in R^m : z_\eta + z_\gamma \leq 1 \ (\eta, \gamma) \in E, \ \alpha_{\eta k} z_\eta + v_k \leq u_k \ (\eta, k) \in F\}$,

where $0 \leq v_k \leq u_k$, E is the set of binary edge inequalities, and F is the set of mixed edge inequalities.

A mixed vertex packing relaxation can be obtained by considering pairwise conflicts between binary variables and between binary and continuous variables (Atamturk et al. [2]). To generate the conflict graph, we can start by setting binary variables to 1 and check which binary variables would be fixed to 0 (creating binary edges) and which continuous variables get their upper bounds tightened (generating mixed edges). The weight on a mixed edge reflects the value of decrease in the upper bound of the continuous variable when the binary variable takes on the value 1.

Below, we perform five probing steps systematically on the binary variables and list the associated edges created in the conflict graph.

1. Let $s_{ij} = 1$ for some i and j. Then $s_{il} = 0$, $\forall l \neq j$ from (11); $\tilde{q}_{ij} \geq u + 1$, [or $\bar{\tilde{q}}_{ij} \leq u - 1$] from (13). It follows that $\tilde{q}_{il} \leq u - \frac{1}{u}$, [or $\bar{\tilde{q}}_{il} \geq u + \frac{1}{u}$] from (14); $y_{il} = 1$, [i.e. $\bar{y}_{il} = 0$] from (15), and $t_l = 1$, [i.e. $\bar{t}_l = 0$] from (10), $\forall l \neq j$.

In addition, in (18) we have $-\tilde{q}_{ij} + \frac{u}{u-1} y_{ij} \leq -u$, or $\bar{\tilde{q}}_{ij} + \frac{u}{u-1} y_{ij} \leq u$, which may tight the bound for $\tilde{q}_{ij}$ ($\geq u + \frac{u}{u-1}$ instead of $u+1$) depending on the value of y_{ij}. Plus, from (12) we get $\bar{\tilde{q}}_{ij} \leq \frac{1}{a_{ij}}(u \sum_{\gamma \in J_i^+} a_{i\gamma} - 1 + \sum_{\gamma \in J_i^-} a_{i\gamma} - (u + \frac{1}{u}) \sum_{\gamma \in J_i^+ - \{j\}} a_{i\gamma}) = u - \frac{1}{a_{ij}}(1 - \sum_{\gamma \in J_i^-} a_{i\gamma} + \frac{1}{u} \sum_{\gamma \in J_i^+ - \{j\}} a_{i\gamma})$. The right hand side has to be nonnegative, otherwise we have fixing s_{ij} at 0. So, we add the assumption:

$$\frac{1}{a_{ij}}(1 - \sum_{\gamma \in J_i^-} a_{i\gamma} + \frac{1}{u} \sum_{\gamma \in J_i^+ - \{j\}} a_{i\gamma}) \leq u.$$

It is stronger than the assumption about $\sum_{\gamma \in J_i^+} s_{i\gamma} = 1$ in the previous section (observable in the form $u \sum_{\gamma \in J_i^+} a_{i\gamma} - (u + \frac{1}{u}) \sum_{\gamma \in J_i^+ - \{j\}} a_{i\gamma} \geq 1 - \sum_{\gamma \in J_i^-} a_{i\gamma}$ versus $u \sum_{\gamma \in J_i^+} a_{i\gamma} \geq 1 - \sum_{\gamma \in J_i^-} a_{i\gamma}$).

Combining two bounds for $\bar{\tilde{q}}_{ij}$, we have $\bar{\tilde{q}}_{ij} \leq min\ (u - 1, u - \frac{1}{a_{ij}}(1 - \sum_{\gamma \in J_i^-} a_{i\gamma} + \frac{1}{u} \sum_{\gamma \in J_i^+ - \{j\}} a_{i\gamma})) = u - max\ (\frac{1}{a_{ij}}(1 - \sum_{\gamma \in J_i^-} a_{i\gamma} + \frac{1}{u} \sum_{\gamma \in J_i^+ - \{j\}} a_{i\gamma}), 1)$. Also, $\bar{\tilde{q}}_{il} \leq \frac{1}{a_{il}}(u \sum_{\gamma \in J_i^+} a_{i\gamma} - 1 + \sum_{\gamma \in J_i^-} a_{i\gamma} - (u + \frac{1}{u}) \sum_{\gamma \in J_i^+ - \{j,l\}} a_{i\gamma}) = u - \frac{1}{a_{il}}(1 - \sum_{\gamma \in J_i^-} a_{i\gamma} + \frac{1}{u} \sum_{\gamma \in J_i^+ - \{j,l\}} a_{i\gamma} - u a_{ij})$, $\forall l \neq j$. This bound becomes redundant and is substituted by $\bar{\tilde{q}}_{il} \leq 2u$ in the case of large a_{ij}.

Besides, we need $\frac{1}{a_{il}}(1 - \sum_{\gamma \in J_i^-} a_{i\gamma} + \frac{1}{u} \sum_{\gamma \in J_i^+ - \{j,l\}} a_{i\gamma} - u a_{ij}) \leq u$, but it is already satisfied by the previous assumption (observable by rewriting $1 - \sum_{\gamma \in J_i^-} a_{i\gamma} + \frac{1}{u} \sum_{\gamma \in J_i^+ - \{j,l\}} a_{i\gamma} - u a_{ij} = 1 - \sum_{\gamma \in J_i^-} a_{i\gamma} - u a_{ij} + \frac{1}{u} \sum_{\gamma \in J_i^+ - \{j\}} a_{i\gamma} - \frac{1}{u} a_{il} < u a_{il}$; the right hand side is positive, and the left hand side is negative taking into account the assumption).

It is necessary to keep $\tilde{q}_{il} \leq u - \frac{1}{u}$, i.e. $\frac{1}{a_{il}}(1 - \sum_{\gamma \in J_i^-} a_{i\gamma} + \frac{1}{u}\sum_{\gamma \in J_i^+ - \{j,l\}} a_{i\gamma} - ua_{ij}) + \frac{1}{u} \leq 0$, which is also satisfied from the same assumption (as we again expand $\frac{1}{u}\sum_{\gamma \in J_i^+ - \{j,l\}} a_{i\gamma} = \frac{1}{u}\sum_{\gamma \in J_i^+ - \{j\}} a_{i\gamma} - \frac{1}{u}a_{il}$). Denote for future use

$$W_{ij} \equiv 1 - \sum_{\gamma \in J_i^-} a_{i\gamma} + \frac{1}{u} \sum_{\gamma \in J_i^+ - \{j\}} a_{i\gamma}.$$

In the conflict graph, binary edges connect s_{ij} with s_{il}, $\bar{y}_{il}$, $\bar{t}_l$, $\forall l \neq j$; and mixed edges connect s_{ij} with $\bar{\tilde{q}}_{ij}$ (the weight $= \max(u+1,\ u+\frac{W_{ij}}{a_{ij}})$), $\tilde{q}_{il}$ (the weight $= u+\frac{1}{u}$), $\bar{\tilde{q}}_{il}$ (the weight $= (u - \frac{1}{u} + \frac{1}{a_{il}}(W_{ij} - ua_{ij}))^+$, where $A^+ = max(A, 0)$), $\forall l \neq j$.

2. Let $s_{ij} = 0$ [i.e. $\bar{s}_{ij} = 1$] for some i and j. Then $\tilde{q}_{ij} \leq u - \frac{1}{u}$, [or $\bar{\tilde{q}}_{ij} \geq u + \frac{1}{u}$] from (14); $y_{ij} = 1$, [i.e. $\bar{y}_{ij} = 0$] from (15), and $t_j = 1$, [i.e. $\bar{t}_j = 0$] from (10). In the conflict graph, the edges connect $\bar{s}_{ij}$ with $\bar{y}_{ij}$, $\bar{t}_j$, and $\tilde{q}_{ij}$ (the weight $= u + \frac{1}{u}$).

3. Let $y_{ij} = 1$ [i.e. $\bar{y}_{ij} = 0$] for some i and j. Then $t_j = 1$, [i.e. $\bar{t}_j = 0$] from (10), thus creating an edge between y_{ij} and $\bar{t}_j$ in the conflict graph. From (18): $-\tilde{q}_{ij} + \frac{u^2}{u-1}s_{ij} \leq 0$, or $\bar{\tilde{q}}_{ij} + \frac{u^2}{u-1}s_{ij} \leq 2u$. Depending on the value of s_{ij}, there is a possibility for a tighter bound for $\tilde{q}_{ij}$ ($\geq \frac{u^2}{u-1}$).

4. Let $y_{ij} = 0$ [i.e. $\bar{y}_{ij} = 1$] for some i and j. Then $s_{ij} = 1$ [i.e. $\bar{s}_{ij} = 0$] from (15), $\tilde{q}_{ij} = u + 1$, [or $\bar{\tilde{q}}_{ij} = u - 1$] from (16) and (17); $s_{il} = 0$, $\forall l \neq j$ from (11). It follows that $\tilde{q}_{il} \leq u - \frac{1}{u}$, [or $\bar{\tilde{q}}_{il} \geq u + \frac{1}{u}$] from (14); $y_{il} = 1$, [i.e. $\bar{y}_{il} = 0$] from (15), and $t_l = 1$, [i.e. $\bar{t}_l = 0$] from (10), $\forall l \neq j$. Also, $\bar{\tilde{q}}_{il} \leq \frac{1}{a_{il}}(u\sum_{\gamma \in J_i^+} a_{i\gamma} - 1 + \sum_{\gamma \in J_i^-} a_{i\gamma} - (u + \frac{1}{u})\sum_{\gamma \in J_i^+ - \{j,l\}} a_{i\gamma} - (u-1)a_{ij}) = u - \frac{1}{a_{il}}(1 - \sum_{\gamma \in J_i^-} a_{i\gamma} + \frac{1}{u}\sum_{\gamma \in J_i^+ - \{j,l\}} a_{i\gamma} - a_{ij})$, or $\tilde{q}_{il} \geq u + \frac{1}{a_{il}}(1 - \sum_{\gamma \in J_i^-} a_{i\gamma} + \frac{1}{u}\sum_{\gamma \in J_i^+ - \{j,l\}} a_{i\gamma} - a_{ij})$, $\forall l \neq j$.

Similarly to #1, we apply bounds 0 and $2u$ in the case of large a_{ij}; it is necessary to have $\frac{1}{a_{il}}(1 - \sum_{\gamma \in J_i^-} a_{i\gamma} + \frac{1}{u}\sum_{\gamma \in J_i^+ - \{j,l\}} a_{i\gamma} - a_{ij}) \leq u$. It is also necessary to keep $\tilde{q}_{il} \leq u - \frac{1}{u}$, i.e. $\frac{1}{a_{il}}(1 - \sum_{\gamma \in J_i^-} a_{i\gamma} + \frac{1}{u}\sum_{\gamma \in J_i^+ - \{j,l\}} a_{i\gamma} - a_{ij}) + \frac{1}{u} \leq 0$.

After expanding $\frac{1}{u}\sum_{\gamma \in J_i^+ - \{j,l\}} a_{i\gamma} = \frac{1}{u}\sum_{\gamma \in J_i^+ - \{j\}} a_{i\gamma} - \frac{1}{u}a_{il}$, we introduce the assumption

$$1 - \sum_{\gamma \in J_i^-} a_{i\gamma} + \frac{1}{u} \sum_{\gamma \in J_i^+ - \{j\}} a_{i\gamma} \leq a_{ij}$$

Otherwise, $y_{ij} = 1$) would be fixed at 1. The current requirement can be written in the form $W_{ij} \leq a_{ij}$, which is stronger than $W_{ij} \leq ua_{ij}$ in #1 (for avoiding fixing s_{ij} at 0.

In the conflict graph, binary edges connect $\bar{y}_{ij}$ with $\bar{s}_{ij}$, s_{il}, $\bar{y}_{il}$, $\bar{t}_l$,$\forall l \neq j$; and mixed edges connect $\bar{y}_{ij}$ with $\tilde{q}_{ij}$ (weight = u-1), $\bar{\tilde{q}}_{ij}$ (weight = u+1), $\tilde{q}_{il}$ (weight = $u + \frac{1}{u}$), $\bar{\tilde{q}}_{il}$ (weight = $(u - \frac{1}{u} + \frac{1}{a_{il}}(W_{ij} - a_{ij}))^+$, where $A^+ = max(A, 0)$), $\forall l \neq j$.

5. Suppose that $t_j = 0$ [i.e. $\bar{t}_j = 1$] for some j. Then $y_{ij} = 0\ \forall i$ from (10); $s_{ij} = 1$ [i.e. $\bar{s}_{ij} = 0$] $\forall i$ from (15); $\tilde{q}_{ij} = u + 1$, [or $\bar{\tilde{q}}_{ij} = u - 1$] $\forall i$ from (16) and (17); $s_{il} = 0$, $\forall i, \forall l \neq j$ from (11). It follows that $\tilde{q}_{il} \leq u - \frac{1}{u}$, [or $\bar{\tilde{q}}_{il} \geq u + \frac{1}{u}$] from (14); $y_{il} = 1$, [i.e. $\bar{y}_{il} = 0$] from (15), and $t_l = 1$, [i.e. $\bar{t}_l = 0$] from (10), $\forall i, \forall l \neq j$. Further, everything related to $\bar{\tilde{q}}_{ij}$ and analyzed above in # 4 can be applied here with addition "$\forall i$".

In the conflict graph, binary edges connect $\bar{t}_j$ with y_{ij}, $\bar{s}_{ij}$, s_{il}, $\bar{y}_{il}$, $\bar{t}_l$, and mixed edges connect $\bar{t}_j$ with $\tilde{q}_{ij}$ (the weight = u-1), $\bar{\tilde{q}}_{ij}$ (weight = u+1), $\tilde{q}_{il}$ (weight = $u + \frac{1}{u}$), $\bar{\tilde{q}}_{il}$ (weight = $(u - \frac{1}{u} + \frac{1}{a_{il}}(W_{ij} - a_{ij}))^+$, where $A^+ = max(A, 0)$), $\forall i,\ \forall l \neq j$.

As an example, Figure 1 depicts the binary vertices and edges of the mixed conflict graph derived from S for $I_T = 2$, $J = 3$. The double lines represent the connection of original and complementary (barred) variables. Although variables s_{ij} and y_{ij} are not connected by binary edges, they have indirect "hyperedge relations" (dash lines) that will be introduced in the next section.

Figures 2 and 3 supplement the binary part of the mixed conflict graph with continuous vertices $\tilde{q}_{ij}$, and $\bar{\tilde{q}}_{ij}$, respectively, together with their adjacent vertices and edges, where $l \neq j$.

2.4 Polyhedral Analysis.

Without loss of generality, we assume that $|J_i^+| = J$, $\forall i$; first J variables in each term i need to be considered for possible transformations; and no binary variables are fixed at 0 or 1. In other words, we consider the problem with all $J + 3JI_T$ variables.

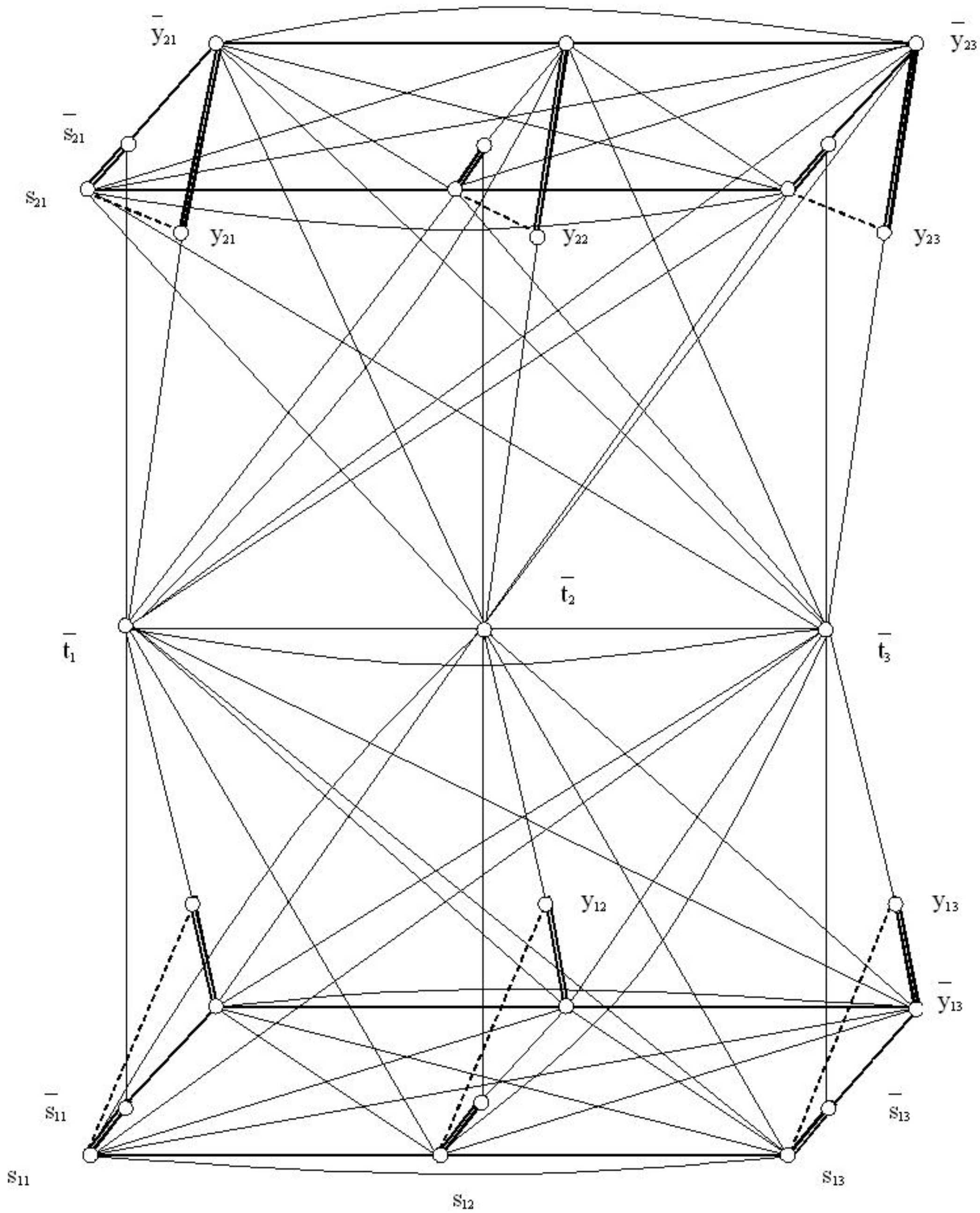

Figure 1: Mixed conflict graph of S: Binary vertices and edges

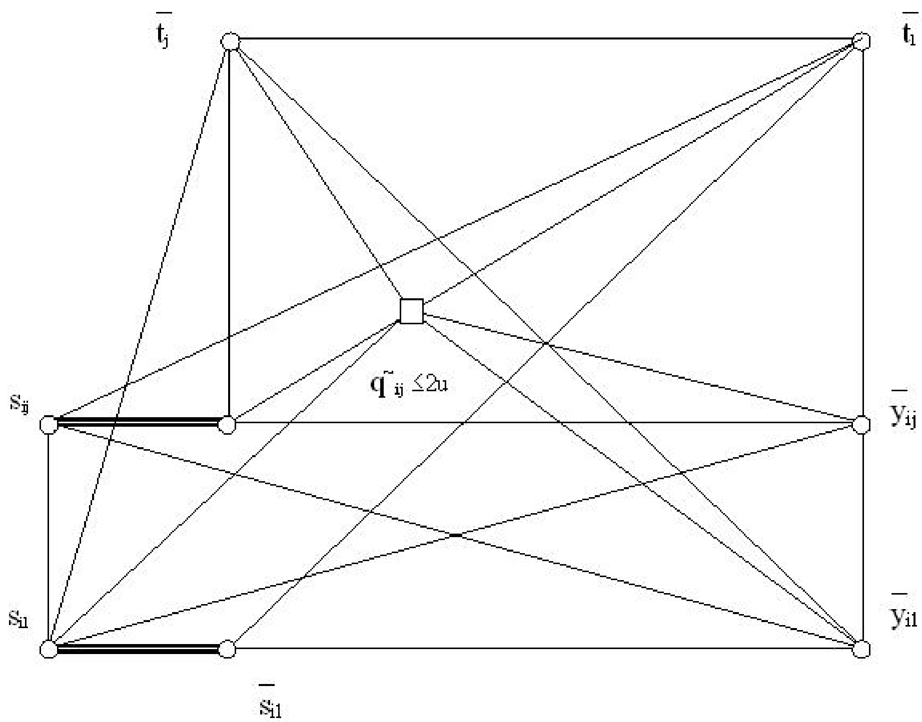

Figure 2: Vertex $\tilde{q}_{ij}$ with adjacent vertices and edges, $l \neq j$

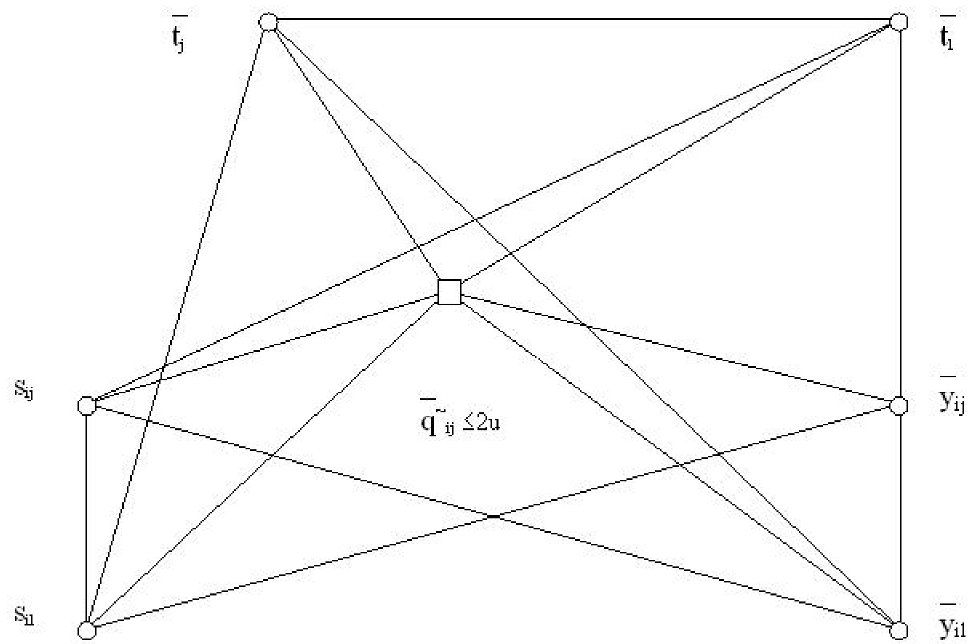

Figure 3: Vertex $\bar{\tilde{q}}_{ij}$ with adjacent vertices and edges, $l \neq j$

Theorem 2.4.1. *conv(S) is full-dimensional.*

Proof. We have the following $J + 3JI_T + 1$ affinely independent points, grouped in 4 sets:

1) JI_T vectors with coordinates: all t =1; all y =1; one $s_{ij} = 1$, all other s =0; $\tilde{q}_{ij} = 2u$ (related to $s_{ij} = 1$), all other $\tilde{q} = u - \frac{1}{u}$;

2) JI_T vectors with coordinates: all t =1; one $y_{ij} = 0$, all other y =1; $s_{ij} = 1$ (related to $y_{ij} = 0$), all other s =0; $\tilde{q}_{ij} = u+1$ (related to $y_{ij} = 0$), all other $\tilde{q} = u - \frac{1}{u}$;

3) J vectors with the coordinates: one $t_j = 0$, all other t =1; $y_{ij} = 0\ \forall i$ (related to $t_j = 0$), all other y =1; $s_{ij} = 1\ \forall i$ (related to $t_j = 0$), all other s =0; $\tilde{q}_{ij} = u + 1\ \forall i$ (related to $t_j = 0$), all other $\tilde{q} = u - \frac{1}{u}$;

4) $1 + JI_T$ vectors with the coordinates: all $s_{ij} = 0$, all other binary variables=1; for $\tilde{q}_{ij}$: the "all zeroes" subvector and unit subvectors of all $\tilde{q}$ multiplied by $u - \frac{1}{u}$. □

Theorem 2.4.2. *The following are trivial facet-defining inequalities for conv(S):*

a) $\sum_{j \in J_i^+} s_{ij} \leq 1 \quad \forall i$;

b) $y_{ij} + s_{ij} \geq 1 \quad \forall i, j \in J_i^+$;

c) $y_{ij} \leq t_j$;

d) $t_j \leq 1$.

The respective sets of $J + 3JI_T$ affinely independent points in each case can easily be found.

We use the mixed conflict graph to identify three families of nontrivial valid inequalities, and prove that they are facet-defining for conv(S). To start, we recall two important mixed conflict graph structures and their respective valid inequalities for the mixed vertex packing polytope (Atamturk et al. [2]).

Definition 2.4.1. *A* star *of vertex k (related to a continuous variable) is a subgraph consisting of vertices k and $T \subseteq N(k)$ (where $N(k)$ denotes the index set of binary vertices adjacent to k) and the edges between k and T.*

The *star inequality* is of the form $\sum_{\eta \in T} \bar{\alpha}_{\eta k} z_\eta + v_k \leq u_k$ where $T = \{\eta_1, \eta_2, ...\eta_t\}$; $\alpha_{\eta_{\gamma-1}k} \leq \alpha_{\eta_\gamma k}$ for $\gamma = 2, ...t$; $\bar{\alpha}_{\eta_1 k} = \alpha_{\eta_1 k}$, $\bar{\alpha}_{\eta_\gamma k} = \alpha_{\eta_\gamma k} - \alpha_{\eta_{\gamma-1}k}$, $\gamma = 2, 3...t$.

Definition 2.4.2. *A* mixed clique *is a subgraph with one vertex related to a continuous variable, adjacent vertices representing a clique of binary variables, and the edges connecting them.*

The *mixed clique inequality* is of the form $\sum_{\eta \in K \subseteq N(k)} \alpha_{\eta k} z_\eta + v_k \leq u_k$.

We can observe a family of mixed clique inequalities in the form of

$$\tilde{q}_{ij} + (u-1)\bar{y}_{ij} + (u + \frac{1}{u})\bar{s}_{ij} \leq 2u,$$

where the edges between $\bar{y}_{ij}$ and $\tilde{q}_{ij}$ have weights $= u - 1$ and the edges between $\bar{s}_{ij}$ and $\tilde{q}_{ij}$ have weights $=(u + \frac{1}{u})$, and prove formally the following result.

Theorem 2.4.3. *The following inequalities are facet-defining for conv(S)*

$$\tilde{q}_{ij} - (u-1)y_{ij} - (u + \frac{1}{u})s_{ij} \leq \frac{u-1}{u} \quad \forall i, j \in J_i^+$$

Proof. First, validity is straightforward.

For $s_{ij} = 0$ and $y_{ij} = 1$, $\tilde{q}_{ij} \leq \frac{u-1}{u} + u - 1 = u - \frac{1}{u}$, which corresponds to $q_{ij} < 0$ (from Table 1).

For $s_{ij} = 1$ and $y_{ij} = 0$, $\tilde{q}_{ij} \leq \frac{u-1}{u} + u + \frac{1}{u} = u + 1$, which corresponds to $q_{ij} = 1$ (from Table 1).

For $s_{ij} = 1$ and $y_{ij} = 1$, $\tilde{q}_{ij} \leq \frac{u-1}{u} + u - 1 + u + \frac{1}{u} = 2u$.

To prove facet defining, we have the following $J + 3JI_T$ affinely independent points (where i* or j* denotes one index in consideration respectively among i and j), satisfying the inequality at equality, i.e. $\tilde{q}_{i^*j^*} - (u-1)y_{i^*j^*} - (u + \frac{1}{u})s_{i^*j^*} =$

$\frac{u-1}{u}$. Specifically, $J + 2JI_T$ vectors from sets 1-3 in the proof of Theorem 2.4.1; and JI_T vectors with the coordinates: all $s_{ij} = 0$, all other binary variables=1; for $\tilde{q}_{ij}$: $u - \frac{1}{u}$ multiplied by: the unit subvector of current $\tilde{q}_{i^*j^*}$, and the sum of the unit subvector of $\tilde{q}_{i^*j^*}$ and the unit subvector of other $\tilde{q}_{ij}, i \neq i^*, j \neq j^*$, i.e. $(u - \frac{1}{u})e_{\tilde{q}_{i^*j^*}}, (u - \frac{1}{u})(e_{\tilde{q}_{i^*j^*}} + e_{\tilde{q}_{ij}}), i \neq i^*, j \neq j^*$. □

One illustrative example for $I_T = 2,\ J = 3,\ i^* = 1,\ j^* = 1$ is summarized in Table 2.

Table 2: Affinely independent vectors in Theorem 2.4.3

t_1	t_2	t_3	y_{11}	y_{12}	y_{13}	y_{21}	y_{22}	y_{23}	s_{11}	s_{12}	s_{13}	s_{21}	s_{22}	s_{23}	$\tilde{q}_{11}$	$\tilde{q}_{12}$	$\tilde{q}_{13}$	$\tilde{q}_{21}$	$\tilde{q}_{22}$	$\tilde{q}_{23}$
1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	2u	$u-\frac{1}{u}$	$u-\frac{1}{u}$	$u-\frac{1}{u}$	$u-\frac{1}{u}$	$u-\frac{1}{u}$
1	1	1	1	1	1	1	1	1	0	1	0	0	0	0	$u-\frac{1}{u}$	2u	$u-\frac{1}{u}$	$u-\frac{1}{u}$	$u-\frac{1}{u}$	$u-\frac{1}{u}$
1	1	1	1	1	1	1	1	1	0	0	1	0	0	0	$u-\frac{1}{u}$	$u-\frac{1}{u}$	2u	$u-\frac{1}{u}$	$u-\frac{1}{u}$	$u-\frac{1}{u}$
1	1	1	1	1	1	1	1	1	0	0	0	1	0	0	$u-\frac{1}{u}$	$u-\frac{1}{u}$	$u-\frac{1}{u}$	2u	$u-\frac{1}{u}$	$u-\frac{1}{u}$
1	1	1	1	1	1	1	1	1	0	0	0	0	1	0	$u-\frac{1}{u}$	$u-\frac{1}{u}$	$u-\frac{1}{u}$	$u-\frac{1}{u}$	2u	$u-\frac{1}{u}$
1	1	1	1	1	1	1	1	1	0	0	0	0	0	1	$u-\frac{1}{u}$	$u-\frac{1}{u}$	$u-\frac{1}{u}$	$u-\frac{1}{u}$	$u-\frac{1}{u}$	2u
1	1	1	0	1	1	1	1	1	1	0	0	0	0	0	u+1	$u-\frac{1}{u}$	$u-\frac{1}{u}$	$u-\frac{1}{u}$	$u-\frac{1}{u}$	$u-\frac{1}{u}$
1	1	1	1	0	1	1	1	1	0	1	0	0	0	0	$u-\frac{1}{u}$	u+1	$u-\frac{1}{u}$	$u-\frac{1}{u}$	$u-\frac{1}{u}$	$u-\frac{1}{u}$
1	1	1	1	1	0	1	1	1	0	0	1	0	0	0	$u-\frac{1}{u}$	$u-\frac{1}{u}$	u+1	$u-\frac{1}{u}$	$u-\frac{1}{u}$	$u-\frac{1}{u}$
1	1	1	1	1	1	0	1	1	0	0	0	1	0	0	$u-\frac{1}{u}$	$u-\frac{1}{u}$	$u-\frac{1}{u}$	u+1	$u-\frac{1}{u}$	$u-\frac{1}{u}$
1	1	1	1	1	1	1	0	1	0	0	0	0	1	0	$u-\frac{1}{u}$	$u-\frac{1}{u}$	$u-\frac{1}{u}$	$u-\frac{1}{u}$	u+1	$u-\frac{1}{u}$
1	1	1	1	1	1	1	1	0	0	0	0	0	0	1	$u-\frac{1}{u}$	$u-\frac{1}{u}$	$u-\frac{1}{u}$	$u-\frac{1}{u}$	$u-\frac{1}{u}$	u+1
0	1	1	0	1	1	0	1	1	1	0	0	1	0	0	u+1	$u-\frac{1}{u}$	$u-\frac{1}{u}$	u+1	$u-\frac{1}{u}$	$u-\frac{1}{u}$
1	0	1	1	0	1	1	0	1	0	1	0	0	1	0	$u-\frac{1}{u}$	u+1	$u-\frac{1}{u}$	$u-\frac{1}{u}$	u+1	$u-\frac{1}{u}$
1	1	0	1	1	0	1	1	0	0	0	1	0	0	1	$u-\frac{1}{u}$	$u-\frac{1}{u}$	u+1	$u-\frac{1}{u}$	$u-\frac{1}{u}$	u+1
1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	$u-\frac{1}{u}$	0	0	0	0	0
1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	$u-\frac{1}{u}$	$u-\frac{1}{u}$	0	0	0	0
1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	$u-\frac{1}{u}$	0	$u-\frac{1}{u}$	0	0	0
1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	$u-\frac{1}{u}$	0	0	$u-\frac{1}{u}$	0	0
1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	$u-\frac{1}{u}$	0	0	0	$u-\frac{1}{u}$	0
1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	$u-\frac{1}{u}$	0	0	0	0	$u-\frac{1}{u}$

Now we provide a new result beyond the described mixed conflict graph. The presence of inequalities (18) and their analysis in the graph construction motivate us to introduce the mixed hyperedges representing the tight relationships among $y_{ij}, s_{ij}, \tilde{q}_{ij}$ (or their complementary variables with bars).

Definition 2.4.3. *A* mixed hyperedge *is a set of at least three vertices, representing at least one binary and at least one continuous variables, not necessarily connected by edges pairwisely.*

Definition 2.4.4. *A* mixed hypergraph *is a mixed conflict graph with mixed hyperedges.*

Recall the concept of a hypergraph as a generalization of a graph for the binary case (see e.g. Easton et al. [13]). Suppose that a graph G consists of a finite set of vertices $V(G)$ and edges $E(G)$. The elements in $E(G)$ are subsets of $V(G)$ of size 2. In a hypergraph H, the set of edges $E(H)$ is formed from the power set of vertices $V(H)$ and each edge has cardinality ≥ 2 (with $= 2$ for the generic graph G). Our definition of the mixed hypergraph is mild in the sense that it does not require special structural properties, which is subject of future research.

The following theorem and illustrating numerical example introduce the *mixed star-clique inequalities*, where we can combine two binary variables $\bar{y}$ and s together with one continuous variable $\bar{\tilde{q}}$ into one hyperedge. The inside variables have the star structure (i.e. the "leading" variable s keeps the mixed edge weight, and the "following" variable $\bar{y}$ takes the difference between the weight of the leading variable and own mixed edge weight as the coefficient in the inequality); plus, hyperedges create the clique of the fixed size. The continuous variable is added as it is, and its upper bound goes to the right hand side of the inequality. The weight of the edge connecting s_{ij} with $\bar{\tilde{q}}_{ij}$ is strengthened to $\max(\frac{u^2}{u-1},\ u+\frac{W_{ij}}{a_{ij}})$ because $\frac{u^2}{u-1} = u + \frac{u}{u-1} > u+1$.

Theorem 2.4.4. *The following inequalities are facet-defining for conv(S)*
$max(\frac{u^2}{u-1},\ u+\frac{W_{ij}}{a_{ij}})s_{ij} + (u+1-max(\frac{u^2}{u-1},\ u+\frac{W_{ij}}{a_{ij}}))(1-y_{ij}) + \sum_{l\in J_i^+-\{j\}}(u-\frac{1}{u}+\frac{1}{a_{ij}}(W_{il}-ua_{il}))^+ s_{il} + ((u-\frac{1}{u}+\frac{1}{a_{ij}}(W_{il}-a_{il}))^+ - (u-\frac{1}{u}+\frac{1}{a_{ij}}(W_{il}-ua_{il}))^+)(1-y_{il}) - \tilde{q}_{ij} \leq 0 \quad \forall i,j \in J_i^+$

Proof. First, validity of the inequalities follows from the description of probing and graph construction above.

Further, we have the following $J + 3JI_T$ affinely independent points (where i* or j* denotes one index in consideration respectively among i and j), satisfying the inequality at equality, i.e.

$$max(\frac{u^2}{u-1},\ u+\frac{W_{i^*j^*}}{a_{i^*j^*}})s_{i^*j^*}+(u+1-max(\frac{u^2}{u-1},\ u+\frac{W_{i^*j^*}}{a_{i^*j^*}}))(1-y_{i^*j^*})+\sum_{l\in J^+_{i^*}-\{j^*\}}(u-\frac{1}{u}+\frac{1}{a_{i^*j^*}}(W_{i^*l}-ua_{i^*l}))^+s_{i^*l}+((u-\frac{1}{u}+\frac{1}{a_{i^*j^*}}(W_{i^*l}-a_{i^*l}))^+-(u-\frac{1}{u}+\frac{1}{a_{i^*j^*}}(W_{i^*l}-ua_{i^*l}))^+)(1-y_{i^*l})-\tilde{q}_{i^*j^*}=0$$

1) JI_T vectors with the coordinates: see set 1 in Theorem 2.4.1 with the exception for $\tilde{q}_{i^*j^*}$: $\tilde{q}_{i^*j^*}=max(\frac{u^2}{u-1},\ u+\frac{W_{i^*j^*}}{a_{i^*j^*}})$ (related to $s_{i^*j^*}=1$); or $\tilde{q}_{i^*j^*}=(u-\frac{1}{u}+\frac{1}{a_{i^*j^*}}(W_{i^*l}-ua_{i^*l}))^+$ (related to $s_{i^*l}=1,\ l\neq j^*$); or $\tilde{q}_{i^*j^*}=0$ (related to $s_{ij}=1,\ i\neq i^*$).

2) JI_T vectors with the coordinates: see set 2 in Theorem 2.4.1 with the exception for $\tilde{q}_{i^*j^*}$: $\tilde{q}_{i^*j^*}=u+1$ (related to $y_{i^*j^*}=0$); or $\tilde{q}_{i^*j^*}=(u-\frac{1}{u}+\frac{1}{a_{i^*j^*}}(W_{i^*l}-a_{i^*l}))^+$ (related to $y_{i^*l}=0,\ l\neq j^*$); or $\tilde{q}_{i^*j^*}=0$ (related to $y_{ij}=0,\ i\neq i^*$).

3) J vectors with the coordinates: see set 3 in Theorem 2.4.1 with the exception for $\tilde{q}_{i^*j^*}$: $\tilde{q}_{i^*j^*}=u+1$ (related to $t_{j^*}=0$); or $\tilde{q}_{i^*j^*}=(u-\frac{1}{u}+\frac{1}{a_{i^*j^*}}(W_{i^*l}-a_{i^*l}))^+$ (related to $t_j=0,\ j\neq j^*$).

4) JI_T vectors with the coordinates: see set 4 in Theorem 2.4.1 with the exception for $\tilde{q}_{i^*j^*}$:$\tilde{q}_{i^*j^*}=0$ for all vectors. □

An illustrative example for $I_T=2,\ J=3,\ i^*=1,\ j^*=1$ is in Table 3, where $S_{11}=max(\frac{u^2}{u-1},\ u+\frac{W_{11}}{a_{11}})$, $S_{12}=(u-\frac{1}{u}+\frac{1}{a_{11}}(W_{12}-ua_{12}))^+$, $S_{13}=(u-\frac{1}{u}+\frac{1}{a_{11}}(W_{13}-ua_{13}))^+$, $Y_{12}=(u-\frac{1}{u}+\frac{1}{a_{11}}(W_{12}-a_{12}))^+$, $Y_{13}=(u-\frac{1}{u}+\frac{1}{a_{11}}(W_{13}-a_{13}))^+$.

Example 2.4.1. Consider $I_T=1,\ J=3,\ u=4,\ a_{11}=\frac{7}{2},\ a_{12}=3,\ a_{13}=\frac{11}{4}$.

The corresponding polytope is

$y_{11}-t_1\leq 0$

$y_{12}-t_2\leq 0$

$y_{13}-t_3\leq 0$

Table 3: Affinely independent vectors in Theorem 2.4.4

t_1	t_2	t_3	y_{11}	y_{12}	y_{13}	y_{21}	y_{22}	y_{23}	s_{11}	s_{12}	s_{13}	s_{21}	s_{22}	s_{23}	$\tilde{q}_{11}$	$\tilde{q}_{12}$	$\tilde{q}_{13}$	$\tilde{q}_{21}$	$\tilde{q}_{22}$	$\tilde{q}_{23}$
1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	S_{11}	$u-\frac{1}{u}$	$u-\frac{1}{u}$	$u-\frac{1}{u}$	$u-\frac{1}{u}$	$u-\frac{1}{u}$
1	1	1	1	1	1	1	1	1	0	1	0	0	0	0	S_{12}	2u	$u-\frac{1}{u}$	$u-\frac{1}{u}$	$u-\frac{1}{u}$	$u-\frac{1}{u}$
1	1	1	1	1	1	1	1	1	0	0	1	0	0	0	S_{13}	$u-\frac{1}{u}$	2u	$u-\frac{1}{u}$	$u-\frac{1}{u}$	$u-\frac{1}{u}$
1	1	1	1	1	1	1	1	1	0	0	0	1	0	0	0	$u-\frac{1}{u}$	$u-\frac{1}{u}$	2u	$u-\frac{1}{u}$	$u-\frac{1}{u}$
1	1	1	1	1	1	1	1	1	0	0	0	0	1	0	0	$u-\frac{1}{u}$	$u-\frac{1}{u}$	$u-\frac{1}{u}$	2u	$u-\frac{1}{u}$
1	1	1	1	1	1	1	1	1	0	0	0	0	0	1	0	$u-\frac{1}{u}$	$u-\frac{1}{u}$	$u-\frac{1}{u}$	$u-\frac{1}{u}$	2u
1	1	1	0	1	1	1	1	1	1	0	0	0	0	0	u+1	$u-\frac{1}{u}$	$u-\frac{1}{u}$	$u-\frac{1}{u}$	$u-\frac{1}{u}$	$u-\frac{1}{u}$
1	1	1	1	0	1	1	1	1	0	1	0	0	0	0	Y_{12}	u+1	$u-\frac{1}{u}$	$u-\frac{1}{u}$	$u-\frac{1}{u}$	$u-\frac{1}{u}$
1	1	1	1	1	0	1	1	1	0	0	1	0	0	0	Y_{13}	$u-\frac{1}{u}$	u+1	$u-\frac{1}{u}$	$u-\frac{1}{u}$	$u-\frac{1}{u}$
1	1	1	1	1	1	0	1	1	0	0	0	1	0	0	0	$u-\frac{1}{u}$	$u-\frac{1}{u}$	u+1	$u-\frac{1}{u}$	$u-\frac{1}{u}$
1	1	1	1	1	1	1	0	1	0	0	0	0	1	0	0	$u-\frac{1}{u}$	$u-\frac{1}{u}$	$u-\frac{1}{u}$	u+1	$u-\frac{1}{u}$
1	1	1	1	1	1	1	1	0	0	0	0	0	0	1	0	$u-\frac{1}{u}$	$u-\frac{1}{u}$	$u-\frac{1}{u}$	$u-\frac{1}{u}$	u+1
0	1	1	0	1	1	0	1	1	1	0	0	1	0	0	u+1	$u-\frac{1}{u}$	$u-\frac{1}{u}$	u+1	$u-\frac{1}{u}$	$u-\frac{1}{u}$
1	0	1	1	0	1	1	0	1	0	1	0	0	1	0	Y_{12}	u+1	$u-\frac{1}{u}$	$u-\frac{1}{u}$	u+1	$u-\frac{1}{u}$
1	1	0	1	1	0	1	1	0	0	0	1	0	0	1	Y_{13}	$u-\frac{1}{u}$	u+1	$u-\frac{1}{u}$	$u-\frac{1}{u}$	u+1
1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	$u-\frac{1}{u}$	0	0	0	0
1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	$u-\frac{1}{u}$	0	0	0
1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	$u-\frac{1}{u}$	0	0
1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	$u-\frac{1}{u}$	0
1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	$u-\frac{1}{u}$

$s_{11} + s_{12} + s_{13} \leq 1$

$-\frac{7}{2}\tilde{q}_{11} - 3\tilde{q}_{12} - \frac{11}{4}\tilde{q}_{13} + 38s_{11} + 38s_{12} + 38s_{13} \leq 0$

$-\tilde{q}_{11} + 5s_{11} \leq 0$

$-\tilde{q}_{12} + 5s_{12} \leq 0$

$-\tilde{q}_{13} + 5s_{13} \leq 0$

$-\tilde{q}_{11} - \frac{17}{4}s_{11} \leq \frac{15}{4}$

$-\tilde{q}_{12} - \frac{17}{4}s_{12} \leq \frac{15}{4}$

$-\tilde{q}_{13} - \frac{17}{4}s_{13} \leq \frac{15}{4}$

$y_{11} + s_{11} \geq 1$

$y_{12} + s_{12} \geq 1$

$y_{13} + s_{13} \geq 1$

$-\tilde{q}_{11} - 5y_{11} \leq -5$

$-\tilde{q}_{12} - 5y_{12} \leq -5$

$-\tilde{q}_{13} - 5y_{13} \leq -5$

$\tilde{q}_{11} - 3y_{11} \leq 5$

$\tilde{q}_{12} - 3y_{12} \leq 5$

$\tilde{q}_{13} - 3y_{13} \leq 5$

$-3\tilde{q}_{11} + 4y_{11} + 16s_{11} \leq 4$

$-3\tilde{q}_{12} + 4y_{12} + 16s_{12} \leq 4$

$-3\tilde{q}_{13} + 4y_{13} + 16s_{13} \leq 4$

$t_1,\ t_2,\ t_3,\ y_{11},\ y_{12},\ y_{13},\ s_{11},\ s_{12},\ s_{13} \in \{0,1\}$

$\tilde{q}_{11},\ \tilde{q}_{12},\ \tilde{q}_{13} \in [0;8]$

One facet-defining inequality for conv(S) is $-\frac{1}{3}\bar{y}_{11} + \frac{18}{7}\bar{y}_{12} + \frac{33}{14}\bar{y}_{13} + \frac{16}{3}s_{11} + \frac{59}{56}s_{12} + \frac{19}{14}s_{13} + \bar{\tilde{q}}_{11} \leq 8$, where the coefficients come from the weights on the graph (see Figure 4) and calculated as: $\frac{18}{7} = \frac{29}{8} - \frac{59}{56}$, and $\frac{33}{14} = \frac{26}{7} - \frac{19}{14}$.

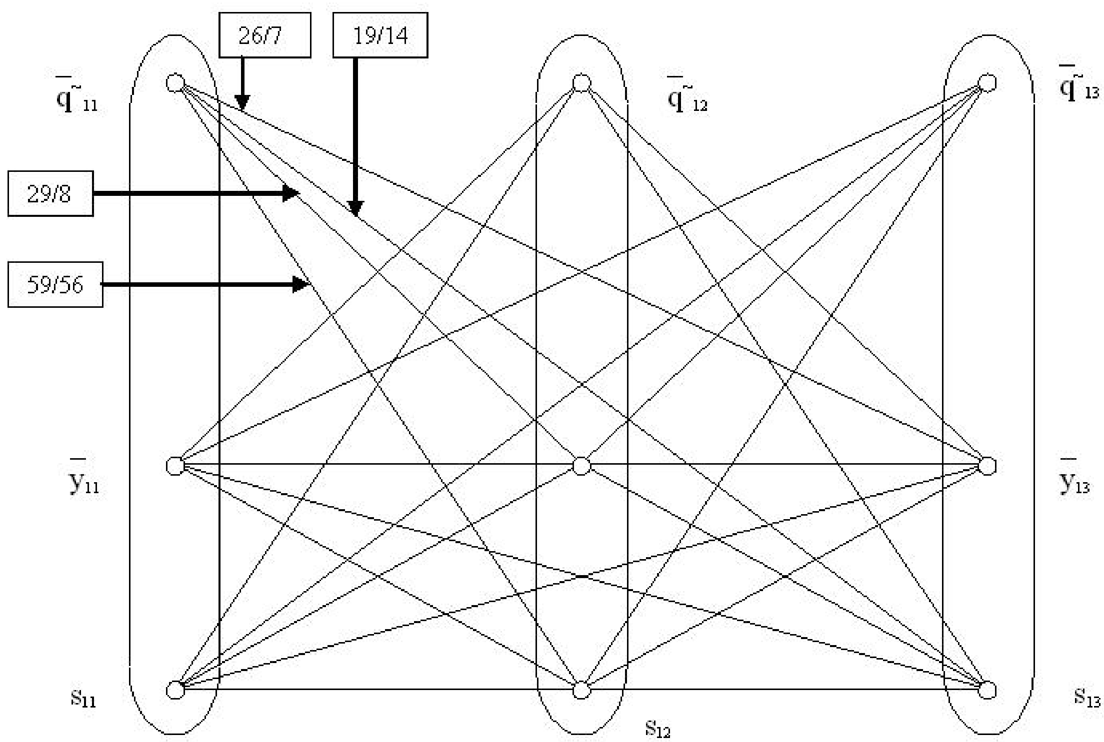

Figure 4: Graph illustrating an example of a mixed star-clique inequality

One more family of facet-defining inequalities can be determined in the form of ”weighted complementary” to the previous ones. We call them *weighted complementary inequalities* because index j in expressions is substituted by ”summations of all other without j”.

Theorem 2.4.5. *The following inequalities are facet-defining for conv(S)* $(u - \frac{1}{u} + \frac{W_{ij} - ua_{ij}}{\sum_{\gamma \in J_i^+ - \{j\}} a_{i\gamma}})^+ s_{ij} + (u - \frac{1}{u} + \frac{W_{ij} - a_{ij}}{\sum_{\gamma \in J_i^+ - \{j\}} a_{i\gamma}} - (u - \frac{1}{u} + \frac{W_{ij} - ua_{ij}}{\sum_{\gamma \in J_i^+ - \{j\}} a_{i\gamma}})^+)(1 - y_{ij})$
$+(u - \frac{1}{u} + \frac{W_{ij} + \frac{1}{u} a_{ij}}{\sum_{\gamma \in J_i^+ - \{j\}} a_{i\gamma}}) \sum_{l \in J_i^+ - \{j\}} s_{il} - \sum_{l \in J_i^+ - \{j\}} \omega_{il} \tilde{q}_{il} \leq 0 \quad \forall i, j \in J_i^+,$
where $\omega_{il} = \frac{a_{il}}{\sum_{\gamma \in J_i^+ - \{j\}} a_{i\gamma}}$.

Proof. As before, the starred index denotes one index in the respective index set. The validity is observable automatically from the following process of choosing $J + 3JI_T$ affinely independent points for dimensionality. We have 4 vector sets again and binary variables as well as $\tilde{q}_{ij}$ with $i \neq i^* \ \forall j$ in each set are as before. The differences are in $\tilde{q}_{i^*j}$.

Set 1. (JI_T vectors). First, consider the q-subvector related to $s_{i^*j^*} = 1$. Set $\tilde{q}_{i^*j^*} = 2u$. Take $l^* \neq j^*$ in $J_{i^*}^+$ with $\tilde{q}_{i^*l^*} = (u - \frac{1}{u} + \frac{1}{a_{i^*l^*}}(W_{i^*j^*} - ua_{i^*j^*}))^+$. Without loss of generality, take $l^* = j^* + 1$ with the cycling rule: after the last index, consider the first one. If $\tilde{q}_{i^*l^*} > 0$, take the remaining $\tilde{q}_{i^*l} = u - \frac{1}{u}, \ l \neq \{j^*, l^*\}$; and it is easy to show that $\sum_{\gamma \in J_{i^*}^+ - \{j^*\}} a_{i^*\gamma}(u - \frac{1}{u}) + W_{i^*j^*} - ua_{i^*j^*} = \sum_{\gamma \in J_{i^*}^+ - \{j^*\}} a_{i^*\gamma} \tilde{q}_{i^*\gamma}$. Indeed, plug the value $u - \frac{1}{u}$ for all $\tilde{q}_{i^*l}, \ l \neq \{j^*, l^*\}$ and simplify: the right hand side $= a_{i^*l^*} \tilde{q}_{i^*l^*}$ and the left hand side $= a_{i^*l^*}(u - \frac{1}{u}) + W_{i^*j^*} - ua_{i^*j^*}$, confirming the expression for $\tilde{q}_{i^*l^*} > 0$. If $\tilde{q}_{i^*l^*} = 0$, we try next $\tilde{q}_{i^*l^*+1}$ to be positive, again with the remaining coordinates $= u - \frac{1}{u}$. The value of $\tilde{q}_{i^*l^*+1} > 0$ satisfies the same equality, now in the form of $a_{i^*l^*}(u - \frac{1}{u}) + a_{i^*l^*+1}(u - \frac{1}{u}) + W_{i^*j^*} - ua_{i^*j^*} = a_{i^*l^*+1} \tilde{q}_{i^*l^*+1}$. Thus, $\tilde{q}_{i^*l^*+1} = u - \frac{1}{u} + \frac{1}{a_{i^*l^*+1}}(W_{i^*j^*} - ua_{i^*j^*}) + \frac{a_{i^*l^*}}{a_{i^*l^*+1}}(u - \frac{1}{u})$, if it is positive. Otherwise, we write $\tilde{q}_{i^*l^*+1} = (...)^+$, i.e. $\tilde{q}_{i^*l^*+1} = 0$ and repeat this procedure for $\tilde{q}_{i^*l^*+2}$ and so on until the first success with the positive value or conclude that one does not exist and we are satisfied with zero values of all $\tilde{q}_{i^*l}, \ l \neq j^*$, which means that $a_{i^*j^*}$ is large and the s_{ij}-coefficient in the theorem inequality $(u - \frac{1}{u} + \frac{W_{ij} - ua_{ij}}{\sum_{\gamma \in J_i^+ - \{j\}} a_{i\gamma}})^+ = 0$.
Next, consider the q-subvector related to some $s_{i^*l^*} = 1, \ l^* \neq j^*$. Our task is to choose some q-subvector to convert the "cover" $\sum_{\gamma \in J_{i^*}^+ - \{j^*\}} a_{i^*\gamma} \tilde{q}_{i^*\gamma} \geq \sum_{\gamma \in J_{i^*}^+ - \{j^*\}} a_{i^*\gamma}(u - \frac{1}{u}) + W_{i^*j^*} + \frac{1}{u} a_{i^*j^*}$ into equality subject to famous restrictions: $\tilde{q}_{i^*l^*} \in [\frac{u^2}{u-1}, 2u]$, other

$\tilde{q} \in [0, u - \frac{1}{u}]$. We take $q_{i^*j^*} = u - \frac{1}{u}$ and $q_{i^*l^*} = \frac{u^2}{u-1}$ (which is also $= u + \frac{u}{u-1}$ and $= u + 1 + \frac{1}{u-1}$). Like in the previous case, we pick without loss of generality $\tilde{q}_{i^*l^*+1}$ separately and try to fix the remaining $\tilde{q}_{i^*l},\ l \neq \{j^*, l^*, l^*+1\}$ to $u - \frac{1}{u}$. We would have $a_{i^*l^*}(u - \frac{1}{u}) + a_{i^*l^*+1}(u - \frac{1}{u}) + W_{i^*j^*} + \frac{1}{u}a_{i^*j^*} = a_{i^*l^*}(u + \frac{u}{u-1}) + a_{i^*l^*+1}\tilde{q}_{i^*l^*+1}$. Thus, $\tilde{q}_{i^*l^*+1} = u - \frac{1}{u} + \frac{1}{a_{i^*l^*+1}}(W_{i^*j^*} + \frac{1}{u}a_{i^*j^*}) - \frac{a_{i^*l^*}}{a_{i^*l^*+1}}(\frac{1}{u} + \frac{u}{u-1})$, which is also can be written without j* in the form $\tilde{q}_{i^*l^*+1} = u - \frac{1}{u} + \frac{1}{a_{i^*l^*+1}}(W_{i^*l^*} - a_{i^*l^*}) - \frac{a_{i^*l^*}}{a_{i^*l^*+1}}\frac{1}{u-1}$. If $\tilde{q}_{i^*l^*+1} > 0$, we are done with the subvector. Otherwise, we assign $\tilde{q}_{i^*l^*+1} = 0$ and try $\tilde{q}_{i^*l^*+2}$. Now we would have one positive term more with, therefore, more chances to get the positive value: $a_{i^*l^*}(u - \frac{1}{u}) + a_{i^*l^*+1}(u - \frac{1}{u}) + a_{i^*l^*+2}(u - \frac{1}{u}) + W_{i^*j^*} + \frac{1}{u}a_{i^*j^*} = a_{i^*l^*}(u + \frac{u}{u-1}) + a_{i^*l^*+2}\tilde{q}_{i^*l^*+2}$. Thus, $\tilde{q}_{i^*l^*+2} = u - \frac{1}{u} + \frac{a_{i^*l^*+1}}{a_{i^*l^*+2}}(u - \frac{1}{u}) + \frac{1}{a_{i^*l^*+2}}(W_{i^*j^*} + \frac{1}{u}a_{i^*j^*}) - \frac{a_{i^*l^*}}{a_{i^*l^*+2}}(\frac{1}{u} + \frac{u}{u-1})$, or without j*: $\tilde{q}_{i^*l^*+2} = u - \frac{1}{u} + \frac{1}{a_{i^*l^*+2}}(W_{i^*l^*} - a_{i^*l^*}) - \frac{a_{i^*l^*}}{a_{i^*l^*+2}}\frac{1}{u-1} + \frac{a_{i^*l^*+1}}{a_{i^*l^*+2}}(u - \frac{1}{u})$. We continue this procedure until the first success with the positive value or we are satisfied with the accumulation of all possible terms on the left hand side: $\sum_{\gamma \in J^+_{i^*} - \{j^*\}} a_{i^*\gamma}(u - \frac{1}{u}) + W_{i^*j^*} + \frac{1}{u}a_{i^*j^*} = a_{i^*l^*}(u + \frac{u}{u-1})$.

In addition, $\tilde{q}_{i^*j} = 0$ related to $s_{ij} = 1,\ i \neq i^*\ \forall j$.

Set 2. (JI_T vectors). First, consider the q-subvector related to $y_{i^*j^*} = 0$. Automatically, it means $s_{i^*j^*} = 1$ and $\tilde{q}_{i^*j^*} = u + 1$. Comparing to the first case in Set 1, we have $W_{i^*j^*} - a_{i^*j^*}$ instead of $W_{i^*j^*} - ua_{i^*j^*}$. Take $l^* \neq j^*$ in $J^+_{i^*}$ with $\tilde{q}_{i^*l^*} = u - \frac{1}{u} + \frac{1}{a_{i^*l^*}}(W_{i^*j^*} - a_{i^*j^*})$. Without loss of generality, take $l^* = j^* + 1$ with the cycling rule, and the remaining $\tilde{q}_{i^*l} = u - \frac{1}{u},\ l \neq \{j^*, l^*\}$.

Next, consider the q-subvector related to some $y_{i^*l^*} = 0,\ l^* \neq j^*$. Automatically, it means $s_{i^*l^*} = 1$ and $\tilde{q}_{i^*l^*} = u+1$. We take $q_{i^*j^*} = u - \frac{1}{u}$, pick without loss of generality $\tilde{q}_{i^*l^*+1}$ separately and try to fix the remaining $\tilde{q}_{i^*l},\ l \neq \{j^*, l^*, l^*+1\}$ to $u - \frac{1}{u}$. We would have $a_{i^*l^*}(u - \frac{1}{u}) + a_{i^*l^*+1}(u - \frac{1}{u}) + W_{i^*j^*} + \frac{1}{u}a_{i^*j^*} = a_{i^*l^*}(u+1) + a_{i^*l^*+1}\tilde{q}_{i^*l^*+1}$. Thus, $\tilde{q}_{i^*l^*+1} = u - \frac{1}{u} + \frac{1}{a_{i^*l^*+1}}(W_{i^*j^*} + \frac{1}{u}a_{i^*j^*}) - \frac{a_{i^*l^*}}{a_{i^*l^*+1}}(\frac{1}{u} + 1)$, which is also can be written without j* in the form $\tilde{q}_{i^*l^*+1} = u - \frac{1}{u} + \frac{1}{a_{i^*l^*+1}}(W_{i^*l^*} - a_{i^*l^*})$ and rings a bell.

In addition, $\tilde{q}_{i^*j} = 0$ related to $y_{ij} = 0,\ i \neq i^*\ \forall j$.

Set 3. (J vectors). Like in the previous theorems, this vector set is an abridged version of Set 2 with the propagation of $\tilde{q} = u + 1$ related to $t = 0$.

Set 4.(JI_T vectors). Here we can indicate two points for $\tilde{q}_{i^*j}$ $\forall j$: the "all zeroes" subvector $\tilde{q}$ and the unit subvector of $\tilde{q}_{i^*j^*}$ multiplied by $(u - \frac{1}{u})$. To get $J-2$ vectors more, we return to Set 2, part 1. Take without loss of generality $l^*+1 \neq j^*$ in $J^+_{i^*}$ with $\tilde{q}_{i^*l^*+1} = u - \frac{1}{u} + \frac{1}{a_{i^*l^*+1}}(W_{i^*j^*} - a_{i^*j^*})$ and the remaining $\tilde{q}_{i^*l} = u - \frac{1}{u},\ l \neq \{j^*, l^*+1\}$. After that take $l^* + 2 \neq j^*$ and so on.

For $i \neq i^*$, we can just copy values from the previous theorem.

□

An illustrative example for $I_T = 2,\ J = 3,\ i^* = 1,\ j^* = 1$ is in Table 4, where, according to the description in the proof, (...) is related to the respective expression with $W - ua$ (including possible 0) or $u - \frac{1}{u}$

Table 4: Affinely independent vectors in Theorem 2.4.5

t_1	t_2	t_3	y_{11}	y_{12}	y_{13}	y_{21}	y_{22}	y_{23}	s_{11}	s_{12}	s_{13}	s_{21}	s_{22}	s_{23}	$\tilde{q}_{11}$	$\tilde{q}_{12}$	$\tilde{q}_{13}$	$\tilde{q}_{21}$	$\tilde{q}_{22}$	$\tilde{q}_{23}$
1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	$2u$	(S_{21})	$(\ldots)$	$u-\frac{1}{u}$	$u-\frac{1}{u}$	$u-\frac{1}{u}$
1	1	1	1	1	1	1	1	1	0	1	0	0	0	0	$u-\frac{1}{u}$	$\frac{u^2}{u-1}$	(S_{32})	$u-\frac{1}{u}$	$u-\frac{1}{u}$	$u-\frac{1}{u}$
1	1	1	1	1	1	1	1	1	0	0	1	0	0	0	$u-\frac{1}{u}$	(S_{23})	$\frac{u^2}{u-1}$	$u-\frac{1}{u}$	$u-\frac{1}{u}$	$u-\frac{1}{u}$
1	1	1	1	1	1	1	1	1	0	0	0	1	0	0	0	0	0	2u	$u-\frac{1}{u}$	$u-\frac{1}{u}$
1	1	1	1	1	1	1	1	1	0	0	0	0	1	0	0	0	0	$u-\frac{1}{u}$	2u	$u-\frac{1}{u}$
1	1	1	1	1	1	1	1	1	0	0	0	0	0	1	0	0	0	$u-\frac{1}{u}$	$u-\frac{1}{u}$	2u
1	1	1	0	1	1	1	1	1	1	0	0	0	0	0	u+1	Y_{21}	$u-\frac{1}{u}$	$u-\frac{1}{u}$	$u-\frac{1}{u}$	$u-\frac{1}{u}$
1	1	1	1	0	1	1	1	1	0	1	0	0	0	0	$u-\frac{1}{u}$	u+1	Y_{32}	$u-\frac{1}{u}$	$u-\frac{1}{u}$	$u-\frac{1}{u}$
1	1	1	1	1	0	1	1	1	0	0	1	0	0	0	$u-\frac{1}{u}$	Y_{23}	u+1	$u-\frac{1}{u}$	$u-\frac{1}{u}$	$u-\frac{1}{u}$
1	1	1	1	1	1	0	1	1	0	0	0	1	0	0	0	0	0	u+1	$u-\frac{1}{u}$	$u-\frac{1}{u}$
1	1	1	1	1	1	1	0	1	0	0	0	0	1	0	0	0	0	$u-\frac{1}{u}$	u+1	$u-\frac{1}{u}$
1	1	1	1	1	1	1	1	0	0	0	0	0	0	1	0	0	0	$u-\frac{1}{u}$	$u-\frac{1}{u}$	u+1
0	1	1	0	1	1	0	1	1	1	0	0	1	0	0	u+1	Y_{21}	$u-\frac{1}{u}$	u+1	$u-\frac{1}{u}$	$u-\frac{1}{u}$
1	0	1	1	0	1	1	0	1	0	1	0	0	1	0	$u-\frac{1}{u}$	u+1	Y_{32}	$u-\frac{1}{u}$	u+1	$u-\frac{1}{u}$
1	1	0	1	1	0	1	1	0	0	0	1	0	0	1	$u-\frac{1}{u}$	Y_{23}	u+1	$u-\frac{1}{u}$	$u-\frac{1}{u}$	u+1
1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	$u-\frac{1}{u}$	0	0	0	0	0
1	1	1	0	1	1	1	1	1	1	0	0	0	0	0	u+1	$u-\frac{1}{u}$	Y_{31}	$u-\frac{1}{u}$	$u-\frac{1}{u}$	$u-\frac{1}{u}$
1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	$u-\frac{1}{u}$	0	0
1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	$u-\frac{1}{u}$	0
1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	$u-\frac{1}{u}$

All three families of nontrivial inequalities are necessary in the description of conv(S).

Example 2.4.1 (continued): $I_T = 1,\ J = 3,\ u = 4,\ a_{11} = \frac{7}{2},\ a_{12} = 3,\ a_{13} = \frac{11}{4}$.

We can derive 3 mixed clique inequalities:

$\tilde{q}_{11} - 3y_{11} - \frac{17}{4}s_{11} \leq \frac{3}{4}$,

$\tilde{q}_{12} - 3y_{12} - \frac{17}{4}s_{12} \leq \frac{3}{4}$,

$\tilde{q}_{13} - 3y_{13} - \frac{17}{4}s_{13} \leq \frac{3}{4}$;

3 mixed star-clique inequalities:

$\frac{16}{3}s_{11} - \frac{1}{3}(1-y_{11}) + \frac{59}{56}s_{12} + \frac{19}{14}s_{13} + \frac{18}{7}(1-y_{12}) + \frac{33}{14}(1-y_{13}) - \tilde{q}_{11} \leq 0$ (mentioned above),

$\frac{16}{3}s_{12} - \frac{1}{3}(1-y_{12}) + 0s_{11} + \frac{23}{24}s_{13} + \frac{163}{48}(1-y_{11}) + \frac{11}{4}(1-y_{13}) - \tilde{q}_{12} \leq 0$,

$\frac{16}{3}s_{13} - \frac{1}{3}(1-y_{13}) + 0s_{11} + \frac{7}{22}s_{12} + \frac{37}{11}(1-y_{11}) + \frac{36}{11}(1-y_{12}) - \tilde{q}_{13} \leq 0$,

and 3 weighted complementary inequalities:

$10s_{11} + \frac{21}{2}(1-y_{11}) + \frac{199}{8}(s_{12}+s_{13}) - 3\tilde{q}_{12} - \frac{11}{4}\tilde{q}_{13} \leq 0$,

$14s_{12} + 9(1-y_{12}) + \frac{107}{4}(s_{11}+s_{13}) - \frac{7}{2}\tilde{q}_{11} - \frac{11}{4}\tilde{q}_{13} \leq 0$,

$16s_{13} + \frac{33}{4}(1-y_{13}) + \frac{443}{16}(s_{11}+s_{12}) - \frac{7}{2}\tilde{q}_{11} - 3\tilde{q}_{12} \leq 0$.

Here are the instances demonstrating that all of them are necessary. We take $t_1 = t_2 = t_3 = 1$ and provide the fractional points in the format $(y_{11}, y_{12}, y_{13}, s_{11}, s_{12}, s_{13}, \tilde{q}_{11}, \tilde{q}_{12}, \tilde{q}_{13})$ which satisfy exactly 8 out of 9 of these inequalities in each case:

point $(1, 1, 1, \frac{21}{37}, 0, 0, \frac{228}{37}, 0, 0)$ is cut off by $10s_{11} + \frac{21}{2}(1-y_{11}) + \frac{199}{8}(s_{12}+s_{13}) - 3\tilde{q}_{12} - \frac{11}{4}\tilde{q}_{13} \leq 0$,

point $(1, 1, 1, \frac{135}{232}, 0, 0, \frac{90}{29}, \frac{15}{4}, 0)$ is cut off by $14s_{12} + 9(1-y_{12}) + \frac{107}{4}(s_{11}+s_{13}) - \frac{7}{2}\tilde{q}_{11} - \frac{11}{4}\tilde{q}_{13} \leq 0$,

point $(1, 1, 1, \frac{495}{928}, 0, 0, \frac{165}{58}, 0, \frac{15}{4})$ is cut off by $16s_{13} + \frac{33}{4}(1-y_{13}) + \frac{443}{16}(s_{11}+s_{12}) - \frac{7}{2}\tilde{q}_{11} - 3\tilde{q}_{12} \leq 0$,

point $(\frac{3}{4}, 1, 1, 1, 0, 0, 5, \frac{15}{4}, \frac{15}{4})$ is cut off by $\frac{16}{3}s_{11} - \frac{1}{3}(1-y_{11}) + \frac{59}{56}s_{12} + \frac{19}{14}s_{13} + \frac{18}{7}(1-y_{12}) + \frac{33}{14}(1-y_{13}) - \tilde{q}_{11} \leq 0$,

point $(\frac{163}{168}, 1, 1, 1, 0, 0, \frac{443}{56}, 0, \frac{15}{4})$ is cut off by $\frac{16}{3}s_{12} - \frac{1}{3}(1-y_{12}) + 0s_{11} + \frac{23}{24}s_{13} + \frac{163}{48}(1-y_{11}) + \frac{11}{4}(1-y_{13}) - \tilde{q}_{12} \leq 0$,

point $(\frac{37}{42}, 1, 1, 1, 0, 0, \frac{107}{14}, \frac{15}{4}, 0)$ is cut off by $\frac{16}{3}s_{13} - \frac{1}{3}(1-y_{13}) + 0s_{11} + \frac{7}{22}s_{12} + \frac{37}{11}(1 -$

$y_{11}) + \frac{36}{11}(1 - y_{12}) - \tilde{q}_{13} \leq 0$,

point $(\frac{12}{29}, 1, 1, \frac{17}{29}, 0, 0, \frac{181}{29}, \frac{15}{4}, \frac{15}{4})$ is cut off by $\tilde{q}_{11} - 3y_{11} - \frac{17}{4}s_{11} \leq \frac{3}{4}$,

point $(1, \frac{12}{29}, 1, 0, \frac{17}{29}, 0, \frac{15}{4}, \frac{181}{29}, \frac{15}{4})$ is cut off by $\tilde{q}_{12} - 3y_{12} - \frac{17}{4}s_{12} \leq \frac{3}{4}$,

point $(1, 1, \frac{12}{29}, 0, 0, \frac{17}{29}, \frac{15}{4}, \frac{15}{4}, \frac{181}{29})$ is cut off by $\tilde{q}_{13} - 3y_{13} - \frac{17}{4}s_{13} \leq \frac{3}{4}$.

2.5 Separation and Computational Issues.

In this section, we describe the separation technique for applying our developed inequalities in a branch and cut framework. Polynomial-time solvability (for exact or heuristic separation algorithms) is always important. Our scheme is based on finding maximal elements in arrays of J variables that can be done in linear time (via $J - 1$ comparisons).

We should note that branching on binary variables s and y can be time-consuming due to the large number of multiple optimal LP relaxation solutions as illustrated in the example below. In addition, in the current implementation, we do not consider cuts with t variables (the variables connecting different "layers") since we do not yet have enough knowledge on them. However, it is possible to perform explicit branching on them or note that they have "correlated behavior" with y-values and thus cuts on y implicitly influence t. In our computations, we focus mainly on cutting planes and evaluate their impact in the solution process.

Example 2.4.1 (continued): $I_T = 1,\ J = 3,\ u = 4,\ a_{11} = \frac{7}{2},\ a_{12} = 3,\ a_{13} = \frac{11}{4}$. The objective value of the LP relaxation is 3.354 with 6 optimal solutions where one coordinate has $t^* = 1$, another $t^* = \frac{12}{29} \approx 0.414$, and another $t^* = \frac{17}{29} \approx 0.586$; the y^*-values are the same as respective t^*; the s^*-values are 0, $\frac{17}{29} \approx 0.586$, $\frac{12}{29} \approx 0.414$ which satisfy $y^* + s^* = 1$ coordinatewise; and the $\tilde{q}^*$-values are respectively $\frac{15}{4} = 3.75$, $\frac{181}{29} \approx 6.241$, and $\frac{639}{116} \approx 5.509$. We add 3 mixed clique inequalities to the root node. This results in MIP optimality with the optimal objective value 3.479 and optimal vectors are presented in Set 3 of the proof for Theorem 2.4.3.

Separation Heuristics

For mixed star-clique inequalities and weighted complementary inequalities we apply separation heuristics. Without loss of generality, we simplify the notations and drop the i index. Consider the inequalities:

$A_j s_j + B_j(1 - y_j) + \sum_{l \neq j}(C_{lj} s_l + D_{lj}(1 - y_l)) - \tilde{q}_j \leq 0$, and

$E_j s_j + F_j(1 - y_j) + G_j \sum_{l \neq j} s_l - \sum_{l \neq j} H_l \tilde{q}_l \leq 0$,

where capital letters can be seen as functions of the respective index, and partition the index set into two parts: one is just index "j" and the other is all $l \neq j$.

Given an LP solution with $(y^*, s^*, \tilde{q}^*)$, for mixed star-clique inequalities we need to find a partition with $A_j s_j^* + B_j(1 - y_j^*) + \sum_{l \neq j}(C_{lj} s_l^* + D_{lj}(1 - y_l^*)) - \tilde{q}_j^* > 0$. For weighted complementary inequalities, we are looking for a partition with $E_j s_j^* + F_j(1 - y_j^*) + G_j \sum_{l \neq j} s_l^* - \sum_{l \neq j} H_l \tilde{q}_l^* > 0$. We extend the classical separation approaches related to covers by Crowder et al [12]. We also apply the famous weakening technique from Van Roy and Wolsey [42]: dropping the plus sign in the $()^+$ expressions. In the case of no necessity to use this technique (i.e. the coefficients are positive), we have sufficient conditions for polynomial-time solvability, as shown below.

Consider a maximal cut violation as a separation problem. The weighted complementary inequalities (after weakening) can be written as:

$a_{ij}(u-1)(1-y_{ij}) + (1 - \sum_{\gamma \in J_i^-} a_{i\gamma} + u \sum_{\gamma \in J_i^+ - \{j\}} a_{i\gamma} + \frac{1}{u} a_{ij}) \sum_{l \in J_i^+ - \{j\}} s_{il} + (1 - \sum_{\gamma \in J_i^-} a_{i\gamma} + u \sum_{\gamma \in J_i^+ - \{j\}} a_{i\gamma} - u a_{ij}) s_{ij} - \sum_{l \in J_i^+ - \{j\}} a_{il} \tilde{q}_{il} \leq 0$

For brevity, we drop the index i, and denote $\Psi = 1 - \sum_{\gamma \in J^-} a_\gamma + u \sum_{\gamma \in J^+} a_\gamma$ (different for each i) the number presented in the formulation of the problem, see constraint (12).

Introducing the characteristic vector z for the yet-to-be-determined partition (j, all other $l \neq j$) in the same manner as in the covering problems (where z is the unit vector for the indicating variable j), we can show that the separation problem for weighted complementary inequalities is polynomial-time solvable. Indeed, the y-term becomes $(u-1)\sum_{\gamma} a_{\gamma} z_{\gamma}(1-y^*_{\gamma})$; the $\tilde{q}$-term is $\sum_{\gamma} a_{\gamma}\tilde{q}^*_{\gamma}(z_{\gamma}-1)$; the s_j-term is $\sum_{\gamma} s^*_{\gamma}(\Psi - 2ua_{\gamma})z_{\gamma}$; the s_l-term can be viewed initially as quadratic with respect to z: $(\Psi - \sum_{\gamma}(u-\frac{1}{u})a_{\gamma}z_{\gamma})\sum_{\gamma} s^*_{\gamma}(1-z_{\gamma})$, but it can be simplified into a linear one combining with s_j-term, and using the facts that $z_{\gamma_1}z_{\gamma_2}=0$ and $z^2_{\gamma}=z_{\gamma}$ in the following way:

$(\Psi - \sum_{\gamma}(u-\frac{1}{u})a_{\gamma}z_{\gamma})\sum_{\gamma} s^*_{\gamma}(1-z_{\gamma}) + \sum_{\gamma} s^*_{\gamma}(\Psi - 2ua_{\gamma})z_{\gamma} = \Psi\sum_{\gamma} s^*_{\gamma} - \Psi\sum_{\gamma} s^*_{\gamma}z_{\gamma} - \sum_{\gamma}(u-\frac{1}{u})a_{\gamma}z_{\gamma}\sum_{\gamma} s^*_{\gamma} + \sum_{\gamma}(u-\frac{1}{u})a_{\gamma}z_{\gamma}\sum_{\gamma} s^*_{\gamma}z_{\gamma} + \Psi\sum_{\gamma} s^*_{\gamma}z_{\gamma} - \sum_{\gamma} 2ua_{\gamma}s^*_{\gamma}z_{\gamma} = \Psi\sum_{\gamma} s^*_{\gamma} - \sum_{\gamma}(u-\frac{1}{u})a_{\gamma}z_{\gamma}\sum_{\gamma} s^*_{\gamma} + \sum_{\gamma}(u-\frac{1}{u})a_{\gamma}s^*_{\gamma}z_{\gamma} - \sum_{\gamma} 2ua_{\gamma}s^*_{\gamma}z_{\gamma} = (\sum_{\gamma} s^*_{\gamma})(\Psi - (u-\frac{1}{u})\sum_{\gamma} a_{\gamma}z_{\gamma}) - (u+\frac{1}{u})\sum_{\gamma} a_{\gamma}z_{\gamma}s^*_{\gamma}$.

The separation objective function (subject to the unit vector z) becomes $max_z(\sum_{\gamma} s^*_{\gamma})(\Psi - (u-\frac{1}{u})\sum_{\gamma} a_{\gamma}z_{\gamma}) - (u+\frac{1}{u})\sum_{\gamma} a_{\gamma}z_{\gamma}s^*_{\gamma} + (u-1)\sum_{\gamma} a_{\gamma}z_{\gamma}(1-y^*_{\gamma}) + \sum_{\gamma} a_{\gamma}\tilde{q}^*_{\gamma}(z_{\gamma}-1)$. The z-coefficient with the maximal value gives j in the partition with the condition that an optimal objective value is positive. So, we are looking for index j^*, if any, which is related to the maximal value ($j^* = argmax_j$) among $-(u+\frac{1}{u})a_j s^*_j + (u-1)a_j(1-y^*_j) + a_j\tilde{q}^*_j - (u-\frac{1}{u})a_j\sum_{\gamma} s^*_{\gamma}$ with the condition that this value $> \sum_{\gamma}(a_{\gamma}\tilde{q}^*_{\gamma} - \Psi s^*_{\gamma})$.

The weakening in this type of inequalities may happen if some a_{ij} bigger than all other entries in the magnitude of the same row in matrix A. Indeed, $u - \frac{1}{u} + \frac{W_{ij}-ua_{ij}}{\sum_{\gamma\in J_i^+-\{j\}} a_{i\gamma}} < 0$ means $a_{ij} > \sum_{\gamma\in J_i^+-\{j\}} a_{i\gamma} + \frac{1}{u}(1-\sum_{\gamma\in J_i^--\{j\}} a_{i\gamma})$. At most one such element can be in the row. Moreover, it can be regulated by choosing the parameter u.

In a similar manner, the mixed star-clique inequalities are polynomial-time separable. We note that we can omit the plus sign in the y-related expressions $(u - \frac{1}{u} + \frac{1}{a_{ij}}(W_{il}-a_{il}))^+$. They are always positive with our assumption that no binary variable is fixed a priori (i.e. $W_{il} \le a_{il}\ \forall l \in J_i^+$) and necessity to satisfy the "cover" (12) with one value $\tilde{q}$ fixed at $u+1$ and all other $q \in [0, u-\frac{1}{u}]$. Indeed, (12) requires

$\sum_{j\in J_i^+} a_{ij}\tilde{q}_{ij} \geq 1 - \sum_{j\in J_i^-} a_{ij} + u\sum_{j\in J_i^+} a_{ij}$. Suppose that $\tilde{q}_{il_1} = u+1$, $\tilde{q}_{il_2} = 0$ (for a future contradiction), and the best choice for all other $\tilde{q} = u - \frac{1}{u}$. We have $(u-\frac{1}{u})\sum_{j\in J_i^+ - \{l_1,\, l_2\}} a_{ij} + (u+1)a_{il_1} \geq 1 - \sum_{j\in J_i^-} a_{ij} + u\sum_{j\in J_i^+} a_{ij}$. So, $a_{il_1} \geq 1 - \sum_{j\in J_i^-} a_{ij} + ua_{il_2} + \frac{1}{u}\sum_{j\in J_i^+ - \{l_1,\, l_2\}} a_{ij} \geq 1 - \sum_{j\in J_i^-} a_{ij} + uW_{il_2} + \frac{1}{u}\sum_{j\in J_i^+ - \{l_1,\, l_2\}} a_{ij} = 1 - \sum_{j\in J_i^-} a_{ij} + u(1 - \sum_{j\in J_i^-} a_{ij}) + u\sum_{j\in J_i^+ - \{l_2\}} a_{ij} + \frac{1}{u}\sum_{j\in J_i^+ - \{l_1,\, l_2\}} a_{ij}$. Thus, $(1-u)a_{il_1} \geq (u+1)(1 - \sum_{j\in J_i^-} a_{ij}) + (u+\frac{1}{u})\sum_{j\in J_i^+ - \{l_1,\, l_2\}} a_{ij}$ with a contradiction because $u > 1$ and the left hand side is negative, but the right hand side is positive.

After weakening, the mixed star-clique inequalities can be written in the form:

$$max(\frac{u^2}{u-1},\ u + \frac{W_{ij}}{a_{ij}})s_{ij} + min(-\frac{1}{u-1},\ 1 - \frac{W_{ij}}{a_{ij}})(1-y_{ij}) +$$

$$\sum_{l\in J_i^+ - \{j\}} ((u - \frac{1}{u} + \frac{1}{a_{ij}}(W_{il} - ua_{il}))s_{il} + (u-1)\frac{a_{il}}{a_{ij}}(1-y_{il})) - \tilde{q}_{ij} \leq 0$$

After multiplying each term by a_{ij} and dropping for brevity the index i, denote (different for each i) $\Phi = 1 - \sum_{\gamma\in J^-} a_\gamma + \frac{1}{u}\sum_{\gamma\in J^+} a_\gamma$. This number is connected with the previous notations as follows: $\Phi = \Psi - (u-\frac{1}{u})\sum_{\gamma\in J^+} a_\gamma$ and $\Phi = W_\gamma + \frac{1}{u}a_\gamma$. The y_l-term becomes $(u-1)\sum_\gamma a_\gamma(1-z_\gamma)(1-y^*_\gamma)$; the y_j-term is $\sum_\gamma min(\frac{-a_\gamma}{u-1}, \frac{u+1}{u}a_\gamma - \Phi)z_\gamma(1-y^*_\gamma)$; the s_j-term is $\sum_\gamma s^*_\gamma z_\gamma max(\frac{u^2}{u-1}a_\gamma, (u-\frac{1}{u})a_\gamma + \Phi)$; the s_l-term can be viewed initially as quadratic with respect to z: $(\Phi + \sum_\gamma(u-\frac{1}{u})a_\gamma z_\gamma)\sum_\gamma s^*_\gamma(1-z_\gamma) + (u+\frac{1}{u})\sum_\gamma a_\gamma s^*_\gamma(z_\gamma - 1)$, but it can be simplified into a linear one combining with s_j-term and using the facts that $z_{\gamma_1}z_{\gamma_2} = 0$ and $z^2_\gamma = z_\gamma$ in the following way:
$(\Phi + \sum_\gamma(u-\frac{1}{u})a_\gamma z_\gamma)\sum_\gamma s^*_\gamma(1-z_\gamma) + (u+\frac{1}{u})\sum_\gamma a_\gamma s^*_\gamma(z_\gamma - 1) + \sum_\gamma s^*_\gamma z_\gamma max(\frac{u^2}{u-1}a_\gamma, (u-\frac{1}{u})a_\gamma + \Phi) = \Phi\sum_\gamma s^*_\gamma - \Phi\sum_\gamma s^*_\gamma z_\gamma + \sum_\gamma(u-\frac{1}{u})a_\gamma z_\gamma\sum_\gamma s^*_\gamma - \sum_\gamma(u-\frac{1}{u})a_\gamma z_\gamma\sum_\gamma s^*_\gamma z_\gamma + (u+\frac{1}{u})\sum_\gamma a_\gamma s^*_\gamma z_\gamma - (u+\frac{1}{u})\sum_\gamma a_\gamma s^*_\gamma + \sum_\gamma s^*_\gamma z_\gamma max(\frac{u^2}{u-1}a_\gamma, (u-\frac{1}{u})a_\gamma + \Phi) = \Phi\sum_\gamma s^*_\gamma - \Phi\sum_\gamma s^*_\gamma z_\gamma + \sum_\gamma(u-\frac{1}{u})a_\gamma z_\gamma\sum_\gamma s^*_\gamma - \sum_\gamma(u-\frac{1}{u})a_\gamma s^*_\gamma z_\gamma + (u+\frac{1}{u})\sum_\gamma a_\gamma s^*_\gamma z_\gamma - (u+\frac{1}{u})\sum_\gamma a_\gamma s^*_\gamma + \sum_\gamma s^*_\gamma z_\gamma max(\frac{u^2}{u-1}a_\gamma, (u-\frac{1}{u})a_\gamma + \Phi) = \Phi\sum_\gamma s^*_\gamma - \Phi\sum_\gamma s^*_\gamma z_\gamma + \sum_\gamma(u-\frac{1}{u})a_\gamma z_\gamma\sum_\gamma s^*_\gamma + \frac{2}{u}\sum_\gamma a_\gamma s^*_\gamma z_\gamma - (u+\frac{1}{u})\sum_\gamma a_\gamma s^*_\gamma + \sum_\gamma s^*_\gamma z_\gamma max(\frac{u^2}{u-1}a_\gamma, (u-\frac{1}{u})a_\gamma + \Phi) = (\sum_\gamma s^*_\gamma)(\Phi + (u-\frac{1}{u})\sum_\gamma a_\gamma z_\gamma) + \sum_\gamma s^*_\gamma z_\gamma(-\Phi + \frac{2}{u}a_\gamma + max(\frac{u^2}{u-1}a_\gamma, (u-\frac{1}{u})a_\gamma + \Phi)) -$

$(u+\frac{1}{u})\sum_\gamma a_\gamma s^*_\gamma = (\sum_\gamma s^*_\gamma)(\Phi + (u-\frac{1}{u})\sum_\gamma a_\gamma z_\gamma) + \sum_\gamma s^*_\gamma z_\gamma max(a_\gamma(\frac{u^2}{u-1}+\frac{2}{u}) - \Phi, (u+\frac{1}{u})a_\gamma) - (u+\frac{1}{u})\sum_\gamma a_\gamma s^*_\gamma$.

The separation objective function (subject to the unit vector z) becomes $max_z(\sum_\gamma s^*_\gamma)(\Phi + (u-\frac{1}{u})\sum_\gamma a_\gamma z_\gamma) + \sum_\gamma s^*_\gamma z_\gamma max(a_\gamma(\frac{u^2}{u-1}+\frac{2}{u}) - \Phi, (u+\frac{1}{u})a_\gamma) - (u+\frac{1}{u})\sum_\gamma a_\gamma s^*_\gamma + (u-1)\sum_\gamma a_\gamma(1-z_\gamma)(1-y^*_\gamma) + \sum_\gamma min(\frac{-a_\gamma}{u-1}, \frac{u+1}{u}a_\gamma - \Phi)z_\gamma(1-y^*_\gamma) - \sum_\gamma a_\gamma \tilde{q}^*_\gamma z_\gamma$. The z-coefficient with the maximal value gives j in the partition with the condition that an optimal objective value is positive. So, we are looking for index j^*, if any, which is related to the maximal value ($j^* = argmax_j$) among $s^*_j max(a_j(\frac{u^2}{u-1}+\frac{2}{u}) - \Phi, (u+\frac{1}{u})a_j) + (1-y^*_j)(-(u-1)a_j + min(\frac{-a_j}{u-1}, \frac{u+1}{u}a_j - \Phi)) - a_j\tilde{q}^*_j + (u-\frac{1}{u})a_j\sum_\gamma s^*_\gamma$ with the condition that this value $> \sum_\gamma((a_\gamma(u+\frac{1}{u}) - \Phi)s^*_\gamma - (u-1)a_\gamma(1-y^*_\gamma))$.

Preliminary experimentation indicates that it is reasonable to add all mixed clique inequalities to the root node. In general, the separation problem for these inequalities is NP-hard (see Atamturk et al. [2]) where for its solution it was suggested that enumeration be used for small graphs since the search for cliques is restricted to adjacent vertices of a single continuous vertex. However, in our particular problem, these inequalities have a fixed small size, do not contain the data a_{ij}, and are not "hard" to generate. Adding them at the root node will strengthen the formulation. Another good alternative is applying the same max-element technique and select one index j (with the largest violation) for each row i.

2.6 Computational Experiments.

We perform groups of computational experiments taking into account the size of the instances, the use of our own preprocessor to fix a part of the variables, dependence on parameter u, the choice of different objective functions, and the use of the strong cutting planes developed (Theorems 2.4.3-2.4.5). Our computational platform is developed in C++. We use IBM/ILOG CPLEX 12.2 with Concert Technology for comparison. The results herein are performed on a desktop with 1.6 GHz Pentium 4

CPU, 1 Gb RAM, running Microsoft Windows XP.

We remark that the full-dimensional analysis (Theorem 2.4.1) serves as foundation for our preprocessor. Thus, in just one pass ($i = 0...I_T - 1$) with internal ($j = 0...J - 1$) we can perform pre-processing and fixing as follows:

1) check the condition for $\sum_{j \in J_i^+} s_{ij} = 0$ with fixing in row i: all $s_{ij} = 0$, $y_{ij} = 1$, $t_j = 1$,

2) check the condition $W_{ij} > ua_{ij}$ (and condition 1 not working) with fixing $s_{ij} = 0$, $y_{ij} = 1$, $t_j = 1$,

3) check the condition $W_{ij} > a_{ij}$ (and condition 2 not working) with fixing $y_{ij} = 1$, $t_j = 1$.

For brevity, we use matrices $A_{I_T J}$ with positive entries only. We use both pseudo-random and deterministic approaches to generate instances of medium to large-scale for benchmark analysis herein.

The experiments are designed as follows.

Experiments 1. We consider large scale instances with the "simple" objective function indicated in the formulation section with the goal to evaluate the usefulness of the mixed-clique cuts. This is motivated by preliminary analysis of Example 2.4.1 and some small instances where it becomes clear that this family of inequalities would play a very important role in the solution process. The inequalities are added explicitly to the problem without activating the user's preprocessor. We run CPLEX using its internal preprocessor and without any user's cutcallbacks. The entries of matrices $A_{I_T J}$ are selected as pseudo-random uniformly distributed floating-point numbers in the narrow range from 5 to 6. They are generated and recorded in a 5-decimal form.

Table 5 shows the results. The first two columns provide the problem size and characteristics. This is followed by the total time elapsed to solve the instances without and with the added mixed clique cuts. The column "Speedup" calculates the improvement in CPU time when cuts are added. We observe good speedup gain as the size of instances increases.

Table 5: Results of Experiments 1

I_T x J	Value of u	Total Time Elapsed		Speedup
		no cuts CPU_1	with mixed cliques CPU_2	CPU1/CPU2
100 x 100	300	97.63 sec	76.09 sec	1.28
100 x 300	390	373.98 sec	280.44 sec	1.33
150 x 300	390	697.75 sec	449.41 sec	1.55
200 x 300	390	1092.27 sec	603.78 sec	1.81
250 x 300	400	1640.30 sec	653.63 sec	2.51
300 x 300	400	2512.73 sec	836.52 sec	3.00

Experiments 2. The q-term in the objective function can be set differently depending on supplemental goals. We modify the q-term in the objective function as

$$min \sum_j t_j + \sum_i \sum_j y_{ij} + \frac{1}{2uI_TJ} \sum_i \sum_j (max_i a_{ij}) \tilde{q}_{ij}$$

(with a possible goal to discourage big positive magnitudes of $a_{ij}\tilde{q}_{ij}$ while considering maximal elements in each row of A). We can observe that this objective function does not produce numerous multiple optimal solutions for the LP relaxation due to the presence of a_{ij}.

Example 2.6.1.

We use the same data as in Example 2.4.1 with the new objective function above. The resulting instance has only one optimal solution of the LP relaxation. Specifically, the optimal objective value is 5.696, $t_1=\frac{12}{29}$, $t_2=\frac{17}{29}$, $t_3=1$, $y_{11}=\frac{12}{29}$, $y_{12}=\frac{17}{29}$, $y_{13}=1$, $s_{11}=\frac{17}{29}$, $s_{12}=\frac{12}{29}$, $s_{13}=0$, $\tilde{q}_{11}=\frac{181}{29}$, $\tilde{q}_{12}=\frac{937}{174}$, $\tilde{q}_{13}=0$. This point can be cut off by either strong inequality listed above: first or second mixed clique, third star-clique, or first or second weighted complementary inequalities. The optimal solution has the objective value 5.767, with $t_1=0$, $t_2=1$, $t_3=1$, $y_{11}=0$, $y_{12}=1$, $y_{13}=1$, $s_{11}=1$, $s_{12}=0$, $s_{13}=0$, $\tilde{q}_{11}=5$, $\tilde{q}_{12}=\frac{15}{4}$, $\tilde{q}_{13}=\frac{37}{11}$.

Testing such instances, CPLEX appreciates cuts from any of our three groups of cutting planes (Theorems 2.4.3-2.4.5). We make one more change to the new objective function by multiplying the y-sum by 0.1. This increases the importance of the t variables.

We create a series of challenging instances with deterministic matrices using the following formula: $a_{ij} = a_{base} - 0.01 * step * i + step * j$ (in the loop $i = 0...I_T - 1$, $j = 0...J - 1$) with $a_{base} = 1.2, step = 0.5$. Such "overlapping" structure of increasing entries in each row is illustrated by the following example:

$$A_{63} = \begin{pmatrix} 1.2 & 1.7 & 2.2 \\ 1.195 & 1.695 & 2.195 \\ 1.19 & 1.69 & 2.19 \\ 1.185 & 1.685 & 2.185 \\ 1.18 & 1.68 & 2.18 \\ 1.175 & 1.675 & 2.175 \end{pmatrix}$$

We run CPLEX in a usual manner with its preprocessor and compare its performance with the situation when CPLEX is enhanced with both our preprocessor and strong cuts. The computational results are summarized in Table 6. They demonstrate the importance of strong cuts in these structured problems. The first three columns have the same meaning as in the previous table and the last two columns show the performance of CPLEX with the usercutbacks and the number of applied user cuts.

Through our computational experiments, the polyhedral results obtained in this chapter are fundamental in solving the difficult MIP instances. We generate instances that are larger than the publicly available ones and demonstrate the solution speed through the addition of strong cuts and pre-processing. We can claim that we are able to solve instances in hundreds for I_T and J dimensions in less than one hour.

Table 6: Results of Experiments 2

I_T x J	u	Total Time Elapsed CPU_1	with user cuts CPU_2	# of user cuts
20 x 10	100	34.31 sec	8.58 sec	274
22 x 11	100	81.22 sec (or 1.35 min)	19.27 sec	369
26 x 13	100	1851.58 sec (or 30.86 min)	25.06 sec	529
32 x 16	100	not solved in 7 hours	29.63 sec	777
40 x 20	100	-	48.64 sec	1233
50 x 25	100	-	113.4 sec (or 1.89 min)	1570
60 x 30	100	-	189.52 sec (or 3.16 min)	2443
80 x 40	100	-	551.02 sec (or 9.18 min)	3979
90 x 45	100	-	938.77 sec (or 15.65 min)	5194
100 x 50	100	-	1553.05 sec (or 25.88 min)	6664

CHAPTER III

ON CONVEXIFICATION OF MINLP CONTAINING SIGNOMIALS

3.1 Overview and Problem Formulation.

This chapter continues convexification and underestimation techniques for nonlinear MIPs. In particular, we consider an optimization problem for choosing variables needed in power transformations for signomial terms. Based on analysis of the polyhedral structure with the use of the conflict graph, the mixed hyperedge method is applied to a new structured problem. New facet-defining inequalities together with their separation are also presented.

We consider the optimization problem:

$$\begin{array}{lll} \min & f(x) & \\ \text{s.t.} & g_k(x) \le 1 & k = 1...m \\ & x > 0 & \text{(and mixed integer in general)} \end{array}$$

where $f(x)$ and $g_k(x)$ have convex terms as well as signomial terms of the form

$$\sum_i c_i \prod_{j=1}^{n} x_j^{a_{ij}}$$

with c_i and $a_{ij} \in R$. In the previous chapter, we provided analysis of the convexification of the Mixed Integer Nonlinear Problem containing terms with $c_i > 0$ (i.e. posynomial terms) plus other functions or terms, which are convex. Similarly, here we pay attention to only signomial terms in $f(x)$ and all $g_k(x)$ that need convexification. Denote I^+ (i.e. I_T in Chapter II) the number of such terms with $c_i > 0$ and respectively I^- for $c_i < 0$.

The problem is written in the form similar to the classical Geometric Programming problem (which is a particular case, where $f(x)$ and all $g_k(x)$ consist of posynomial terms only and the variables are continuous). Without any loss of generality (WLOG) and depending on the solution purposes, special constraints, 0 right hand sides, intervals for the variables and other modifications can be found in the literature (like in e.g. Lundell et al. [31] and references therein).

Recall that the following proposition serves as the main mathematical background of the whole convexification process.

Proposition 3.1.1. *Term $\prod_{j=1}^{n} x_j^{a_j}$ is convex if*

1. *$a_j \leq 0$ for $j = 1...n$ or*
2. *there exists one $a_p > 0$, and all other $a_j \leq 0, j \neq p$ and $\sum_{j=1}^{n} a_j \geq 1$.*

And $\prod_{j=1}^{n} x_j^{a_j}$ is concave (i.e. $-\prod_{j=1}^{n} x_j^{a_j}$ is convex) if $a_j \geq 0$ for $j = 1, ..., n$ and $\sum_{j=1}^{n} a_j \leq 1$.

The last part of this statement defines the conditions for "negative terms" (i.e. those with $c_i < 0$, without taking into account the magnitude of c).

Now we are going to formulate the overall convexification problem being consistent with Chapter II. We introduced

$$y_{ij} = \begin{cases} 1, & \text{if } x_j \text{ in the ith term is transformed} \\ 0, & \text{otherwise} \end{cases}$$

and

$$t_j = \begin{cases} 1, & \text{if } x_j \text{ is transformed in any of the terms where it is found} \\ 0, & \text{otherwise} \end{cases}$$

In addition, we needed

$$s_{ij} = \begin{cases} 1, & \text{if the power of } x_j \text{ in term } i \text{ remains} > 0 \text{ after the transformation} \\ 0, & \quad \text{otherwise} \end{cases}$$

Besides, we used multipliers q_{ij} (bounded by parameter $u > 1$ and their positive shifts $\tilde{q}_{ij} \triangleq q_{ij} + u$) for the powers such that transformed term i becomes $c_i \prod_{j=1}^{n} x_j^{(1-y_{ij})a_{ij}} \tilde{x}_{ij}^{y_{ij}a_{ij}q_{ij}}$. In the case of negative terms we can observe the following relationships in Table 7.

Table 7: a_{ij}, y_{ij}, *and* q_{ij} relations

$a \setminus q$	$q < 0$	$0 < q < 1$	$q = 1$
$a < 0$	$y = 1$	$\times$	$\times$
$a > 0$	$\times$	$y = 1$	$y = 0$
$a = 0$	$\times$	$\times$	$y = 0$

Depending on the initial data a_{ij}, we apply ($y_{ij} = 1$) or do not apply ($y_{ij} = 0$) transformations. Like in Chapter II, we use a "small" value $\frac{1}{u}$, avoiding strict inequalities: $\frac{1}{u} \leq q_{ij} \leq 1$ or $-u \leq q_{ij} \leq -\frac{1}{u}$. We have to put into the constraints the fact that the sum of $a_{ij}q_{ij}$ is at most 1 (in the shifted form: $\sum a_{ij}\tilde{q}_{ij} \leq u \sum_{j=1}^{n} a_{ij} + 1$) as well as the relationships: if $y_{ij} = 0$ then $q_{ij} = 1$, and if $y_{ij} = 1$ then $q_{ij} < 1$, which can be combined (followed from details in Lundell et al. [31]) as $1 - y_{ij} \leq q_{ij} \leq 1 - \frac{1}{u} y_{ij}$. In the shifted form, we have these as the constraints related to $a_{ij} > 0$: $\tilde{q}_{ij} + y_{ij} \geq u + 1$, $\tilde{q}_{ij} + \frac{1}{u} y_{ij} \leq u + 1$, and $\tilde{q}_{ij} \geq u + \frac{1}{u}$.

Denote J_i^-, J_i^0, and J_i^+ the sets of index j in term i for $\{j : a_{ij} < 0\}$, $\{j : a_{ij} = 0\}$, and $\{j : a_{ij} > 0\}$, respectively.

It follows:

$$min \ \sum_{j} t_j + \sum_{i} \sum_{j} y_{ij} + sum \ with \ \tilde{q}_{ij}$$

s.t.

$$\sum_{i=1}^{(I^+ + I^-)} y_{ij} \leq (I^+ + I^-) t_j \quad \forall j \tag{10}$$

$$\sum_{j \in J_i^+} s_{ij} \leq 1 \quad \forall i \in I^+ \tag{11}$$

$$-\sum_{j \in J_i^+} a_{ij} \tilde{q}_{ij} + (-\sum_{j \in J_i^-} a_{ij} + u \sum_{j \in J_i^+} a_{ij} + 1) \sum_{j \in J_i^+} s_{ij} \leq 0 \quad \forall i \in I^+ \tag{12}$$

$$-\tilde{q}_{ij} + (u+1) s_{ij} \leq 0 \quad \forall i \in I^+, j \in J_i^+ \tag{13}$$

$$\tilde{q}_{ij} - (u + \frac{1}{u}) s_{ij} \leq u - \frac{1}{u} \quad \forall i \in I^+, j \in J_i^+ \tag{14}$$

$$y_{ij} + s_{ij} \geq 1 \quad \forall i \in I^+, j \in J_i^+ \tag{15}$$

$$-\tilde{q}_{ij} - (u+1) y_{ij} \leq -(u+1) \quad \forall i \in I^+, j \in J_i^+ \tag{16}$$

$$\tilde{q}_{ij} - (u-1) y_{ij} \leq u+1 \quad \forall i \in I^+, j \in J_i^+ \tag{17}$$

$$-(u-1)\tilde{q}_{ij} + u y_{ij} + u^2 s_{ij} \leq u \quad \forall i \in I^+, j \in J_i^+ \tag{18}$$

$$\sum_{j=1}^{n} a_{ij} \tilde{q}_{ij} \leq u \sum_{j=1}^{n} a_{ij} + 1 \quad \forall i \in I^- \tag{19}$$

$$\tilde{q}_{ij} + y_{ij} \geq u+1 \quad \forall i \in I^-, j \in J_i^+ \tag{20}$$

$$\tilde{q}_{ij} + \frac{1}{u} y_{ij} \leq u+1 \quad \forall i \in I^-, j \in J_i^+ \tag{21}$$

$$\tilde{q}_{ij} \geq u + \frac{1}{u} \quad \forall i \in I^-, j \in J_i^+ \tag{22}$$

$$\tilde{q}_{ij} = u+1, y_{ij} = 0, s_{ij} = 0 \quad \forall i \in I^+, j \in J_i^-, J_i^0 \tag{23}$$

$$y_{ij} = 1, 0 \leq \tilde{q}_{ij} \leq u - \frac{1}{u} \quad \forall i \in I^-, j \in J_i^- \tag{24}$$

$$y_{ij} = 0, \tilde{q}_{ij} = u+1 \quad \forall i \in I^-, j \in J_i^0 \tag{25}$$

$$t, y, s \in \{0,1\}, \tilde{q} \in [0; 2u] \tag{26}$$

One choice for the q-term in the objective function was $-\frac{1}{2uI^+J}\sum_i\sum_j \tilde{q}_{ij}$ in Chapter II (i.e. a scaled averaged $\tilde{q}$-value, where $J = |J_i^+|\ \forall i$ is assumed). Its counterpart for the negative terms can be chosen as $-\frac{1}{(u+1)I^-J}\sum_i\sum_j \tilde{q}_{ij}$. We again emphasize that this choice of the q-term in the objective function is without any loss of generality for our polyhedral analysis.

It is observable that, in fact, our problem contains two autonomous blocks: positive (with $c > 0$, represented by inequalities (11)-(18) in addition to a part of (10) and fixed variables in (23), and analyzed in Chapter II), and negative (with $c < 0$, which we are going to consider now; and for brevity, from now on, we use I for I^-). These two blocks are linked by two items, namely 0-1 variable vector t and numerical parameter $u > 1$. The previous analysis shows that vector t did not play a significant role in the polyhedral results related to the positive block. On the other hand, the choice of parameter u is crucial and, strictly speaking, may depend on the power data a_{ij}. We followed the principle of not interfering optimal solutions a priori. In other words, the numerical value of u solely should not fix any binary variable at values 0 or 1 before solving the program. The same "full-dimensional" strategy is applicable to the negative block with the goal to get some value of u which may be good for the overall problem. However, different u can be used for positive and negative blocks.

As mentioned in Chapter II, aggregated and disaggregated forms of (10) are equivalent, in contrast to the facility location case because we have 0-1 variables y_{ij} instead of continuous ones. Thus, this chapter studies the mixed 0-1 set defined from (10), (19)-(22) rewritten here with omitting the respective index sets (and more details about $a_{ij} > 0$ only in the next section):

$y_{ij} \le t_j$

$\sum_{j=1}^n a_{ij}\tilde{q}_{ij} \le u\sum_{j=1}^n a_{ij} + 1$

$\tilde{q}_{ij} + y_{ij} \ge u + 1$

$\tilde{q}_{ij} + \frac{1}{u}y_{ij} \le u + 1$

$\tilde{q}_{ij} \geq u + \frac{1}{u}$

$\tilde{q}_{ij} \leq u + 1$

$t, y \in \{0, 1\}$

Denote S^- the set defined by these inequalities and conv (S^-) its convex hull.

The outline of the rest of this chapter is as follows. In Section 3.2, we present arguments for the choice of the key parameter u and continue with probing analysis and conflict graph construction to support our main results in Section 3.3. Derived facet-defining inequalities require solving separation problems, and Section 3.4 is dedicated to those. Finally, computational results are presented in Section 3.5.

3.2 Probing and Conflict Graph Construction.

Similarly to Chapter II, we build a conflict graph related to the mixed vertex packing polytope, but it is necessary to justify the choice of parameter u first.

We pay special attention to constraint (19), which does not contain binary variables. Suppose that $a_{ij} > 0$ for all j in term i (or, we can say this interchangeably, "in row i of the matrix of powers A"). The minimal requirement for u is observable from (19) and (22): $\sum_{j \in J_i^+}(u + \frac{1}{u})a_{ij} \leq u \sum_{j \in J_i^+} a_{ij} + 1 \Rightarrow \; u \geq \sum_{j \in J_i^+} a_{ij} \;\; \forall i$ in the row, i.e. $u \geq \max_i \sum_{j \in J_i^+} a_{ij}$.

Next, we need the possibility to have $u + 1$ for each $\tilde{q}_{ij}$. So, $(u+1)a_{ij} + (u + \frac{1}{u}) \sum_{J_i^+ - \{j\}} a_{il} \leq u a_{ij} + u \sum_{l \in J_i^+ - \{j\}} a_{il} + 1 \Rightarrow \; a_{ij} + \frac{1}{u} \sum_{l \in J_i^+ - \{j\}} a_{il} \leq 1 \;\; \forall i, j \in J_i^+$.
Denote

$$\tilde{W}_{ij} = 1 - \frac{1}{u} \sum_{l \in J_i^+ - \{j\}} a_{il}$$

and assume $\tilde{W}_{ij} - a_{ij} \geq 0$ (otherwise, y_{ij} is fixed at 1).

As observable in Table 7 for $a_{ij} \leq 0$, the value of the respective y_{ij} is fixed. So, we have no necessity to deviate from our full-dimensional analysis and it is good enough to consider just $a_{ij} > 0$. For $a_{ij} < 0$, solutions with $\tilde{q}_{ij}$=0 are always feasible. If we would like to have the opportunity of the whole range $0 \leq \tilde{q}_{ij} \leq u - \frac{1}{u}$, we adjust requirements for u in the previous two paragraphs, i.e. $u \geq \max_i \sum_j |a_{ij}|$ for minimal requirement to work with (19), and respectively for the possibility to have $u+1$ for at least one $\tilde{q}_{ij}$ in each row i, adjust $\tilde{W}_{ij} = 1 - \frac{1}{u}\sum_{l \neq j} |a_{ij}|$. So, like in Chapter II, we may just assume that $|J_i^+| = J$, $\forall i$; first J variables in each term i need to be considered for possible transformation; and no binary variables are fixed at 0 or 1. In other words, we consider the problem with $J + 2JI$ variables.

Now we construct the mixed conflict graph, following the same scheme as in Chapter II.

1. Suppose that $y_{ij} = 1$ [i.e. $\bar{y}_{ij} = 0$] for some i and j. Then $t_j = 1$, [i.e. $\bar{t}_j = 0$]; $\tilde{q}_{ij} \leq u + 1 - \frac{1}{u}$ (from (21)); $\bar{\tilde{q}}_{ij} \leq 1 - \frac{1}{u}$ (from (22)), remembering $\bar{\tilde{q}}_{ij} \triangleq u + 1 - \tilde{q}_{ij}$. In the conflict graph, mixed edges connect y_{ij} with $\tilde{q}_{ij}$ (the weight=$\frac{1}{u}$) and $\bar{\tilde{q}}_{ij}$ (the weight=$u + \frac{1}{u}$).

2. Suppose that $y_{ij} = 0$ [i.e. $\bar{y}_{ij} = 1$] for some i and j. Then $\tilde{q}_{ij}$=u+1, $\bar{\tilde{q}}_{ij}$=0 (and the mixed edge weights are 0 and u+1 respectively). Plus, from (19) we have $\tilde{q}_{il} \leq \frac{1}{a_{il}}(u\sum_{\gamma \in J_i^+} a_{i\gamma} + 1 - (u+1)a_{ij} - (u+\frac{1}{u})\sum_{\gamma \in J_i^+ - \{j,l\}} a_{i\gamma}) = u + \frac{1}{u} + \frac{1}{a_{il}}(\tilde{W}_{ij} - a_{ij})\ \forall l \neq j$. Now, $\tilde{q}_{ij} \geq u + \frac{1}{u}$ is automatically satisfied by the assumption above. In addition, $\tilde{q}_{il} \leq min(u+1, u + \frac{1}{u} + \frac{1}{a_{il}}(\tilde{W}_{ij} - a_{ij})) = u + 1 + min(0, \frac{1}{u} - 1 + \frac{1}{a_{il}}(\tilde{W}_{ij} - a_{ij}))$. So, the weight of the edge connecting $\bar{y}_{ij}$ and $\tilde{q}_{il}$ is $(1 - \frac{1}{u} - \frac{1}{a_{il}}(\tilde{W}_{ij} - a_{ij}))^+$. If this value $(...)^+ > 0$, then $y_{il} = 1$ [i.e. $\bar{y}_{ij} = 0$] and the binary edge between $\bar{y}_{ij}$ and $\bar{y}_{il}$, $l \neq j$ appears.

3.3 Polyhedral Analysis.

We use the mixed conflict graph to identify families of nontrivial valid inequalities, and prove that they are facet-defining for conv(S^-). Consider an introductory example first.

Example 3.3.1. $I = 1,\ J = 3,\ u = 4,\ a_{11} = \frac{21}{50},\ a_{12} = \frac{7}{10},\ a_{13} = \frac{2}{5}.$

The corresponding polytope is

$y_{11} - t_1 \leq 0$

$y_{12} - t_2 \leq 0$

$y_{13} - t_3 \leq 0$

$\frac{21}{50}\tilde{q}_{11} + \frac{7}{10}\tilde{q}_{12} + \frac{2}{5}\tilde{q}_{13} \leq \frac{177}{25}$

$\tilde{q}_{11} + y_{11} \geq 5$

$\tilde{q}_{12} + y_{12} \geq 5$

$\tilde{q}_{13} + y_{13} \geq 5$

$\tilde{q}_{11} + \frac{1}{4}y_{11} \leq 5$

$\tilde{q}_{12} + \frac{1}{4}y_{12} \leq 5$

$\tilde{q}_{13} + \frac{1}{4}y_{13} \leq 5$

$t_1,\ t_2,\ t_3,\ y_{11},\ y_{12},\ y_{13} \in \{0, 1\}$

$\tilde{q}_{11},\ \tilde{q}_{12},\ \tilde{q}_{13} \in [\frac{17}{4}; 5]$.

See Figure 5 for a fragment of the conflict graph (without $\bar{\tilde{q}}$ variables). We duplicate the weights of mixed edges in the rectangles above the vertices $\tilde{q}$, where the numbers are related to the respective weights of edges from each vertex $\tilde{q}_{ij}$ to $\bar{y}_{il}$, $l \neq j$ from left to right (missing $\bar{y}_{ij}$) and where the dash is used for 0 (or, in other words, the absence of the edge). From now on, we will use either rectangles or explicit weights on pictures.

If we run the program PORTA [37], we can observe nontrivial facet-defining inequalities like:

a) $-3y_{1j} - 4\tilde{q}_{1j} \leq -20,\ j = 1, 2, 3$;

b) $21y_{11} - 23y_{12} + 84\tilde{q}_{11} \leq 397$;

$-\,9y_{11} + 35y_{12} - 6y_{13} + 140\tilde{q}_{12} \leq 685$;

$-\,21y_{12} + 20y_{13} + 80\tilde{q}_{13} \leq 379$;

c) $-49y_{11} + 14y_{12} + 14y_{13} + 140\tilde{q}_{12} + 80\tilde{q}_{13} \leq 1024$;

$3y_{11} + 3y_{12} - 12y_{13} + 21\tilde{q}_{11} + 35\tilde{q}_{12} \leq 260$;

d) $-63y_{11} - 19y_{12} - 60y_{13} + 140\tilde{q}_{12} \leq 577$

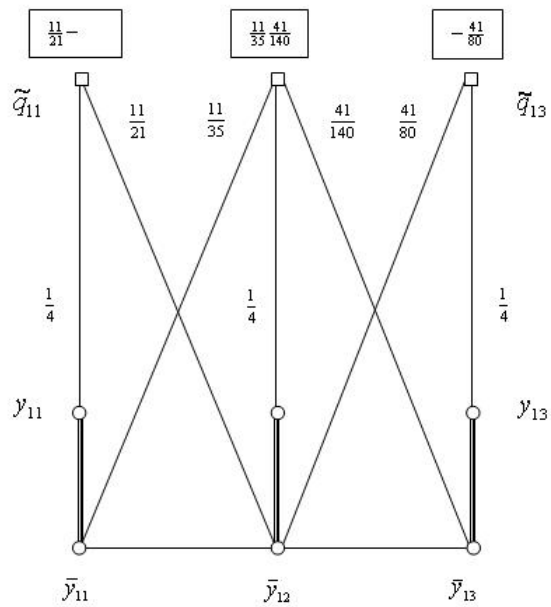

Figure 5: Introductory example

Inequalities similar to (a)-(c) were present in Chapter II, i.e. mixed clique, mixed star-clique, and weighted complementary, respectively. Here we adjust them to the new structure (Theorems 3.3.1-3.3.3). In addition, we introduce a new family, called *incomplete linking inequalities* (Theorems 3.3.4 and 3.3.5), and represented by (d).

The first family is mixed clique inequalities observable from a fragment of the conflict graph of Figure 6 in the form $\bar{\bar{q}}_{ij} + (u + \frac{1}{u})y_{ij} + (u+1)\bar{y}_{ij} \leq\ u+1\ \forall i, j \in J_i^+$.

Theorem 3.3.1. *The following inequalities are facet-defining for conv(S^-)*

$$\tilde{q}_{ij} + \frac{u-1}{u} y_{ij} \geq\ u+1\ \forall i, j \in J_i^+$$

Proof. First, the validness is straightforward from Table 7.

In addition,we have the following $J + 2JI$ affinely independent points (where i* or j* denotes one index in consideration respectively among i and j), satisfying the inequality at equality, i.e. $\tilde{q}_{i^*j^*} + \frac{u-1}{u} y_{i^*j^*} = u+1$, grouped in 3 sets:

1) JI vectors with the coordinates: all t =1; one $y_{ij} = 0$, all other y =1 in row i; $\tilde{q}_{ij} = u+1$ (related to $y_{ij} = 0$), all other $\tilde{q} = u + \frac{1}{u}$;

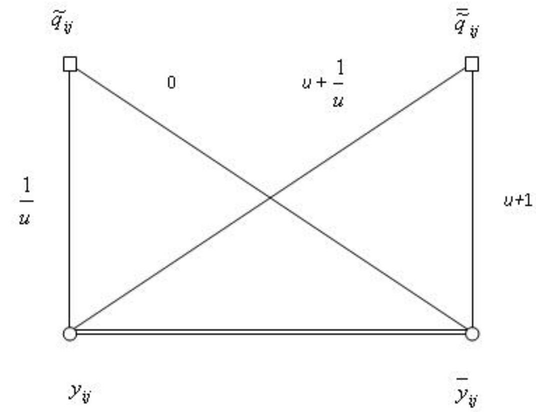

Figure 6: A fragment of the conflict graph illustrating mixed clique inequalities

2) J vectors with the coordinates: one $t_j = 0$, all other t =1 in the row; $y_{ij} = 0\ \forall i$ (related to $t_j = 0$), all other y =1; $\tilde{q}_{ij} = u+1\ \forall i$ (related to $t_j = 0$), all other $\tilde{q} = u+\frac{1}{u}$;

3) JI vectors with the coordinates: all binary variables =1; plus, take q-subvectors with all coordinates $=u + \frac{1}{u}$ and substitute one coordinate (without repetition of the location) by $u+1-\frac{1}{u}$ (except for one vector, related to i^*j^*, in which all q-coordinates remain $u + \frac{1}{u}$). □

One illustrative example for $I = 2,\ J = 3,\ i^* = 1,\ j^* = 1$ is summarized in Table 8.

Table 8: Affinely independent vectors in Theorem 3.3.1

t_1	t_2	t_3	y_{11}	y_{12}	y_{13}	y_{21}	y_{22}	y_{23}	$\tilde{q}_{11}$	$\tilde{q}_{12}$	$\tilde{q}_{13}$	$\tilde{q}_{21}$	$\tilde{q}_{22}$	$\tilde{q}_{23}$
1	1	1	0	1	1	1	1	1	u+1	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$
1	1	1	1	0	1	1	1	1	$u+\frac{1}{u}$	u+1	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$
1	1	1	1	1	0	1	1	1	$u+\frac{1}{u}$	$u+\frac{1}{u}$	u+1	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$
1	1	1	1	1	1	0	1	1	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$	u+1	$u+\frac{1}{u}$	$u+\frac{1}{u}$
1	1	1	1	1	1	1	0	1	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$	u+1	$u+\frac{1}{u}$
1	1	1	1	1	1	1	1	0	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$	u+1
0	1	1	0	1	1	0	1	1	u+1	$u+\frac{1}{u}$	$u+\frac{1}{u}$	u+1	$u+\frac{1}{u}$	$u+\frac{1}{u}$
1	0	1	1	0	1	1	0	1	$u+\frac{1}{u}$	u+1	$u+\frac{1}{u}$	$u+\frac{1}{u}$	u+1	$u+\frac{1}{u}$
1	1	0	1	1	0	1	1	0	$u+\frac{1}{u}$	$u+\frac{1}{u}$	u+1	$u+\frac{1}{u}$	$u+\frac{1}{u}$	u+1
1	1	1	1	1	1	1	1	1	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$
1	1	1	1	1	1	1	1	1	$u+\frac{1}{u}$	$u+1-\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$
1	1	1	1	1	1	1	1	1	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+1-\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$
1	1	1	1	1	1	1	1	1	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+1-\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$
1	1	1	1	1	1	1	1	1	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+1-\frac{1}{u}$	$u+\frac{1}{u}$
1	1	1	1	1	1	1	1	1	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+1-\frac{1}{u}$

In fact, here mixed clique inequalities strengthen inequalities (20), i.e. the coefficient of y_{ij} is $1-\frac{1}{u}$ instead of 1.

The representatives of the next family, mixed star-clique inequalities, were introduced together with our "mixed hyperedge method" in Chapter II. Recall that there we combined two binary variables $\bar{y_{ij}}$ and s_{ij} together with one continuous variable $\bar{\tilde{q}}_{ij}$ into one hyperedge. The inside variables have the star structure (the "leading" variable s_{ij} keeps the mixed edge weight, and the "following" variable $\bar{y_{ij}}$ takes the difference between the weight of the leading variable and own mixed edge weight as the coefficient in the inequality), plus hyperedges create the clique of the fixed size. The continuous variable is added as it is, and its upper bound goes to the right hand side of the inequality.

In our current structure, in spite of not having s-variables, we can apply the same approach. The mixed hyperedges are built by variables $\tilde{q}_{ij}$, y_{ij} and $\bar{y_{il}}$, $l \neq j$. The clique of these hyperedges is built by "automatic" binary edges between y_{ij} and $\bar{y_{ij}}$. (In Chapter II, variables s and $\bar{y}$ played this role of connecting the hyperedges).

Theorem 3.3.2. *The following inequalities are facet-defining for conv(S^-)*

$$\frac{1}{u}y_{ij} + \sum_{l\in J_i^+-\{j\}} (1-\frac{2}{u}-\frac{1}{a_{ij}}(\tilde{W}_{il}-a_{il}))^+(1-y_{il}) + \tilde{q}_{ij} \leq \ u+1 \ \forall i,j \in J_i^+$$

Proof. To prove validness, we firstly can observe that if all $(...)^+ = 0$, the inequality is reduced to (21). In addition, if the value $\tilde{q}_{ij} = u+1$, then $y_{ij} = 0$ and each term in the $(1-y_{il})$-sum is also 0 by the construction of the conflict graph: either $y_{il} = 1$ or the respective weight of the mixed edge has to be 0. Indeed, we cannot have simultaneously $y_{il} = 0$, y_{ij}=0 and $1-\frac{1}{u}-\frac{1}{a_{ij}}(\tilde{W}_{il}-a_{il}) > 0$, $l \neq j$ (y_{ij} would be 1). So, $1-\frac{1}{u}-\frac{1}{a_{ij}}(\tilde{W}_{il}-a_{il}) \leq 0$ and $(1-\frac{1}{u}-\frac{1}{a_{ij}}(\tilde{W}_{il}-a_{il})-\frac{1}{u})^+ = 0$.
The next situation is $\tilde{q}_{ij} \leq u+1-\frac{1}{u}$ with $y_{ij} = 1$. If all $y_{il} = 1$, we are done. So, consider $y_{il^*} = 0$ for some $l^* \neq j$. Let L be the index set of such l^*. We need to show that $\tilde{q}_{ij} \leq u+1-\frac{1}{u}-\sum_{\gamma\in L}(1-\frac{2}{u}-\frac{1}{a_{ij}}(\tilde{W}_{i\gamma}-a_{i\gamma}))^+$. The upper

bound of cardinality $\mid L \mid$ is regulated by inequality (19) and assumptions about u. Thus, WLOG, it is sufficient to consider $\mid L \mid= 1$. Like in the construction of the conflict graph above, we have $\tilde{q}_{ij} \leq u + \frac{1}{u} + \frac{1}{a_{ij}}(\tilde{W}_{il^*} - a_{il^*})$. In fact, $u + \frac{1}{u} \leq \tilde{q}_{ij} \leq min(u+1-\frac{1}{u}, u+\frac{1}{u}+\frac{1}{a_{ij}}(\tilde{W}_{il^*} - a_{il^*})) = u + \frac{1}{u} + min(1-\frac{2}{u}, \frac{1}{a_{ij}}(\tilde{W}_{il^*} - a_{il^*}))$, which is the same as $u + 1 - \frac{1}{u} - (1 - \frac{2}{u} - \frac{1}{a_{ij}}(\tilde{W}_{il^*} - a_{il^*}))^+$, and we are done.

In addition, we have the following $J + 2JI$ affinely independent points (where i* or j* denotes one index in consideration respectively among i and j), satisfying the inequality at equality, i.e. $\frac{1}{u}y_{i^*j^*} + \sum_{l \in J^+_{i^*} - \{j^*\}}(1 - \frac{2}{u} - \frac{1}{a_{i^*j^*}}(\tilde{W}_{i^*l} - a_{i^*l}))^+(1 - y_{i^*l}) + \tilde{q}_{i^*j^*} \leq \ u + 1$:

1) JI vectors with the coordinates: see set 1 in the previous theorem with the exception for $\tilde{q}_{i^*j^*}$ (in J-1 vectors, see the respective tables for the illustrative example): $\tilde{q}_{i^*j^*} = u + 1$ (related to $y_{i^*j^*} = 0$); or $\tilde{q}_{i^*j^*} = u + 1 - \frac{1}{u} - (1 - \frac{2}{u} - \frac{1}{a_{i^*j^*}}(\tilde{W}_{i^*l} - a_{i^*l}))^+$, (related to $y_{i^*l} = 0,\ l \neq j^*$), which is $u + \frac{1}{u} + min(1 - \frac{2}{u}, \frac{1}{a_{i^*j^*}}(\tilde{W}_{i^*l} - a_{i^*l}))$ and becomes $u + \frac{1}{u} + \frac{1}{a_{i^*j^*}}(\tilde{W}_{i^*l} - a_{i^*l})$ if $(...)^+ > 0$;

2) J vectors with the coordinates: see set 2 in the previous theorem with the exception for $\tilde{q}_{i^*j^*}$:

$\tilde{q}_{i^*j^*} = u + 1$ (related to $t_{j^*} = 0$); or $\tilde{q}_{i^*j^*} = u + 1 - \frac{1}{u} - (1 - \frac{2}{u} - \frac{1}{a_{i^*j^*}}(\tilde{W}_{i^*l} - a_{i^*l}))^+$ (related to $t_j = 0,\ j \neq j^*$), which is $u + \frac{1}{u} + min(1 - \frac{2}{u}, \frac{1}{a_{i^*j^*}}(\tilde{W}_{i^*l} - a_{i^*l}))$ and becomes $u + \frac{1}{u} + \frac{1}{a_{i^*j^*}}(\tilde{W}_{i^*l} - a_{i^*l})$ if $(...)^+ > 0$;

3) JI vectors with the coordinates: see set 3 in the previous theorem with the exceptions for $\tilde{q}_{i^*j}$ in J vectors: those $u + 1 - \frac{1}{u}$ in each vector (related to i^*) are substituted by $u + \frac{1}{u} + min(1 - \frac{2}{u}, \frac{1}{a_{i^*l}}(\tilde{W}_{i^*j^*} - a_{i^*j^*}) + \frac{1}{u}\frac{a_{i^*j^*}}{a_{i^*l}})$, plus $\tilde{q}_{i^*j^*} = u + 1 - \frac{1}{u}$.

It is easy to explain the substitution in the last set. We do not assume the guarantee of feasibility of vectors with all binary variables =1, two q-variables $=u + 1 - \frac{1}{u}$ and other$=u + \frac{1}{u}$. So, if we pick some $l \neq j^*$ and plug $q_{i^*j^*} = u + 1 - \frac{1}{u}$ and $q_{i^*\gamma} = u + \frac{1}{u},\ \gamma \neq \{j^*, l\}$ into (19), similarly to the derivations above, we would have $q_{i^*l} \leq u + \frac{1}{u} + \frac{1}{a_{i^*l}}(\tilde{W}_{i^*j^*} - a_{i^*j^*} + \frac{a_{i^*j^*}}{u})$. Also, taking into account the upper bound

$q_{i^*l} \leq u+1-\frac{1}{u}$, we conclude with $u+\frac{1}{u}+min(1-\frac{2}{u}, \frac{1}{a_{i^*l}}(\tilde{W}_{i^*j^*}-a_{i^*j^*})+\frac{1}{u}\frac{a_{i^*j^*}}{a_{i^*l}})$. $\square$

One illustrative example for $I = 2,\ J = 3,\ i^* = 1,\ j^* = 1$ is in Table 9, where $Y_{12} = u+1-\frac{1}{u}-(1-\frac{2}{u}-\frac{1}{a_{11}}(\tilde{W}_{12}-a_{12}))^+$, $Y_{13} = u+1-\frac{1}{u}-(1-\frac{2}{u}-\frac{1}{a_{11}}(\tilde{W}_{13}-a_{13}))^+$, $\tilde{Y}_{12} = u+\frac{1}{u}+min(1-\frac{2}{u}, \frac{1}{a_{12}}(\tilde{W}_{11}-a_{11})+\frac{1}{u}\frac{a_{11}}{a_{12}})$, $\tilde{Y}_{13} = u+\frac{1}{u}+min(1-\frac{2}{u}, \frac{1}{a_{13}}(\tilde{W}_{11}-a_{11})+\frac{1}{u}\frac{a_{11}}{a_{13}})$.

Table 9: Affinely independent vectors in Theorem 3.3.2

t_1	t_2	t_3	y_{11}	y_{12}	y_{13}	y_{21}	y_{22}	y_{23}	$\tilde{q}_{11}$	$\tilde{q}_{12}$	$\tilde{q}_{13}$	$\tilde{q}_{21}$	$\tilde{q}_{22}$	$\tilde{q}_{23}$
1	1	1	0	1	1	1	1	1	u+1	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$
1	1	1	1	0	1	1	1	1	Y_{12}	u+1	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$
1	1	1	1	1	0	1	1	1	Y_{13}	$u+\frac{1}{u}$	u+1	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$
1	1	1	1	1	1	0	1	1	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$	u+1	$u+\frac{1}{u}$	$u+\frac{1}{u}$
1	1	1	1	1	1	1	0	1	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$	u+1	$u+\frac{1}{u}$
1	1	1	1	1	1	1	1	0	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$	u+1
0	1	1	0	1	1	0	1	1	u+1	$u+\frac{1}{u}$	$u+\frac{1}{u}$	u+1	$u+\frac{1}{u}$	$u+\frac{1}{u}$
1	0	1	1	0	1	1	0	1	Y_{12}	u+1	$u+\frac{1}{u}$	$u+\frac{1}{u}$	u+1	$u+\frac{1}{u}$
1	1	0	1	1	0	1	1	0	Y_{13}	$u+\frac{1}{u}$	u+1	$u+\frac{1}{u}$	$u+\frac{1}{u}$	u+1
1	1	1	1	1	1	1	1	1	$u+1-\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$
1	1	1	1	1	1	1	1	1	$u+1-\frac{1}{u}$	$\tilde{Y}_{12}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$
1	1	1	1	1	1	1	1	1	$u+1-\frac{1}{u}$	$u+\frac{1}{u}$	$\tilde{Y}_{13}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$
1	1	1	1	1	1	1	1	1	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+1-\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$
1	1	1	1	1	1	1	1	1	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+1-\frac{1}{u}$	$u+\frac{1}{u}$
1	1	1	1	1	1	1	1	1	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+1-\frac{1}{u}$

Mixed star-clique inequalities do not require any extra conditions and available for all i, j in consideration. If such an inequality has only one $(1-y_{il})$-term (see the first and the third inequalities in part b of Example 3.3.1), it is reduced to a mixed star inequality from Atamturk et al. [2], and if it has no positive $(1-y_{il})$-terms, it becomes inequality (21) in the formulation.

Theorem 3.3.3. *The following inequalities are facet-defining for conv(S^-)*

$$A_{ij}(1-y_{ij}) + B_{ij}\sum_{l\in J_i^+-\{j\}} y_{il} + C_{ij} + \sum_{l\in J_i^+-\{j\}} \omega_{il}\tilde{q}_{il} \leq u+1\ ,$$

for $i, j \in J_i^+$ satisfying all the following conditions:

a) $\frac{\tilde{W}_{ij}-\frac{1}{u}a_{ij}-\frac{1}{u}a_{ir}}{\sum_{\gamma\in J_i^+-\{j\}} a_{i\gamma}} \leq 1-\frac{2}{u}$, *where* a_{ir}=*min* $a_{il},\ l \in J_i^+ - \{j\}$*;*

b) $A_{ij} > 0$*;*

c) $B_{ij} > 0$*;*

d) $\tilde{W}_{ij} > a_{ij}$*,*

where $A_{ij} = 1-\frac{2}{u}-\frac{\tilde{W}_{ij}-a_{ij}}{\sum_{\gamma\in J_i^+-\{j\}} a_{i\gamma}}$, $B_{ij} = -1+\frac{2}{u}+\frac{\tilde{W}_{ij}-\frac{1}{u}a_{ij}}{\sum_{\gamma\in J_i^+-\{j\}} a_{i\gamma}}$, $C_{ij} = \frac{1}{u}+(J-1)(1-\frac{2}{u}-\frac{\tilde{W}_{ij}-\frac{1}{u}a_{ij}}{\sum_{\gamma\in J_i^+-\{j\}} a_{i\gamma}})$, $\omega_{il} = \frac{a_{il}}{\sum_{\gamma\in J_i^+-\{j\}} a_{i\gamma}}$.

Proof. For clarity, we start from expanding the conditions.

a) $\frac{\tilde{W}_{ij}-\frac{1}{u}a_{ij}-\frac{1}{u}a_{ir}}{\sum_{\gamma\in J_i^+-\{j\}}a_{i\gamma}} \le 1-\frac{2}{u}$ (which is also $\frac{\tilde{W}_{ir}-\frac{2}{u}a_{ir}}{\sum_{\gamma\in J_i^+-\{j\}}a_{i\gamma}} \le 1-\frac{2}{u}$) $\iff 1-\frac{1}{u}a_{ij}-\frac{1}{u}a_{ir} \le (1-\frac{1}{u})\sum_{\gamma\in J_i^+-\{j\}}a_{i\gamma} \iff a_{ij} \ge u-a_{ir}-(u-1)\sum_{\gamma\in J_i^+-\{j\}}a_{i\gamma}$;

b) $A_{ij}>0 \iff \frac{\tilde{W}_{ij}-a_{ij}}{\sum_{\gamma\in J_i^+-\{j\}}a_{i\gamma}} < 1-\frac{2}{u} \iff 1-a_{ij} < (1-\frac{1}{u})\sum_{\gamma\in J_i^+-\{j\}}a_{i\gamma} \iff a_{ij} > 1-(1-\frac{1}{u})\sum_{\gamma\in J_i^+-\{j\}}a_{i\gamma}$;

c) $B_{ij}>0 \iff \frac{\tilde{W}_{ij}-\frac{1}{u}a_{ij}}{\sum_{\gamma\in J_i^+-\{j\}}a_{i\gamma}} > 1-\frac{2}{u} \iff 1-\frac{1}{u}a_{ij} > (1-\frac{1}{u})\sum_{\gamma\in J_i^+-\{j\}}a_{i\gamma} \iff a_{ij} < u-(u-1)\sum_{\gamma\in J_i^+-\{j\}}a_{i\gamma}$

As before, the starred index denotes one index in the respective index set. The validity is observable automatically from the following process of choosing $J+2JI$ affinely independent points. We have 3 vector sets again and binary variables as well as $\tilde{q}_{ij}$ with $i \ne i^*\ \forall j$ in each set are as before. The differences are in $\tilde{q}_{i^*j}$ in J vectors of each set (WLOG, first J, see the illustrative example below).

Set 1. (JI vectors).

First, consider the q-subvector related to $y_{i^*j^*}=0$ and other $y_{i^*l}=1,\ l\ne j^*$. It follows: $\tilde{q}_{i^*j^*}=u+1$, other $\tilde{q}\in[u+\frac{1}{u},u+1-\frac{1}{u}]$, or we can write $\tilde{q}_{i^*l}=u+\frac{1}{u}+\Delta_l,\ \Delta_l\in[0,1-\frac{2}{u}],\ l\ne j^*$. After plugging q-values to (19) and simplifying, we have $\sum_{l\in J_{i^*}^+-\{j^*\}}a_{i^*l}\Delta_l \le \tilde{W}_{i^*j^*}-a_{i^*j^*}$. Take $l^*\ne j^*$ in $J_{i^*}^+$; without loss of generality, $l^*=j^*+1$ with the cycling rule: after the last index, consider the first one. Try to assign $\tilde{q}_{i^*l^*}=u+\frac{1}{u}+\frac{1}{a_{i^*l^*}}(W_{i^*j^*}-a_{i^*j^*})$, i.e. $\Delta_{l^*}=\frac{1}{a_{i^*l^*}}(W_{i^*j^*}-a_{i^*j^*})$. If $\tilde{q}_{i^*l^*}<u+1-\frac{1}{u}$, i.e. $\Delta_{l^*}<1-\frac{2}{u}$, keep the remaining $\tilde{q}_{i^*l}=u+\frac{1}{u},\ l\ne\{j^*,l^*\}$; and we are ready to show the equality $A_{i^*j^*}(1-y_{i^*j^*})+B_{i^*j^*}\sum_{l\in J_{i^*}^+-\{j^*\}}y_{i^*l}+C_{i^*j^*}+\sum_{l\in J_{i^*}^+-\{j^*\}}\omega_{i^*l}\tilde{q}_{i^*l}-u-1=0$. Indeed, using $B_{i^*j^*}(J-1)+C_{i^*j^*}=\frac{1}{u}$, we have $(A_{i^*j^*}+B_{i^*j^*}(J-1)+C_{i^*j^*}-u-1)\sum_{\gamma\in J_{i^*}^+-\{j^*\}}a_{i^*\gamma}+\sum_{\gamma\in J_{i^*}^+-\{j^*\}}a_{i^*\gamma}\tilde{q}_{i^*\gamma}=(1-\frac{2}{u}-\frac{\tilde{W}_{i^*j^*}-a_{i^*j^*}}{\sum_{\gamma\in J_{i^*}^+-\{j^*\}}a_{i^*\gamma}}+\frac{1}{u}-u-1)\sum_{\gamma\in J_{i^*}^+-\{j^*\}}a_{i^*\gamma}+(u+\frac{1}{u})\sum_{\gamma\in J_{i^*}^+-\{j^*,l^*\}}a_{i^*\gamma}+a_{i^*l^*}\tilde{q}_{i^*l^*}=-(u+\frac{1}{u})\sum_{\gamma\in J_{i^*}^+-\{j^*\}}a_{i^*\gamma}-(\tilde{W}_{i^*j^*}-a_{i^*j^*})+(u+\frac{1}{u})\sum_{\gamma\in J_{i^*}^+-\{j^*,l^*\}}a_{i^*\gamma}+a_{i^*l^*}\tilde{q}_{i^*l^*}=-(u+\frac{1}{u})a_{i^*l^*}-(\tilde{W}_{i^*j^*}-a_{i^*j^*})+a_{i^*l^*}\tilde{q}_{i^*l^*}=0$, confirming the expression for $\tilde{q}_{i^*l^*}$. Otherwise, set $\Delta_{l^*}=1-\frac{2}{u}$ and assign the excess to the next index, considering $\sum_{l\in J_i^+-\{j,l^*\}}a_{i^*l}\Delta_l \le \tilde{W}_{i^*j}-$

$a_{i^*j^*} - (1-\frac{2}{u})a_{i^*l^*}$. Try $\Delta_{l^*+1} = \frac{1}{a_{i^*l^*+1}}(W_{i^*j^*} - a_{i^*j^*}) - (1-\frac{2}{u})\frac{a_{i^*l^*}}{a_{i^*l^*+1}}$, if positive and $< 1-\frac{2}{u}$. Now, $-(u+\frac{1}{u})\sum_{\gamma\in J^+_{i^*}-\{j^*\}} a_{i^*\gamma} - (\tilde{W}_{i^*j^*} - a_{i^*j^*}) + \sum_{\gamma\in J^+_{i^*}-\{j^*\}} a_{i^*\gamma}\tilde{q}_{i^*\gamma} = -(u+\frac{1}{u})\sum_{\gamma\in J^+_{i^*}-\{j^*\}} a_{i^*\gamma} - (\tilde{W}_{i^*j^*} - a_{i^*j^*}) + (u+\frac{1}{u})\sum_{\gamma\in J^+_{i^*}-\{j^*,l^*,l^*+1\}} a_{i^*\gamma} + (u+1-\frac{1}{u})a_{i^*l^*} + a_{i^*l^*+1}\tilde{q}_{i^*l^*+1} = -(u+\frac{1}{u})a_{i^*l^*+1} - (\tilde{W}_{i^*j^*} - a_{i^*j^*}) + (1-\frac{2}{u})a_{i^*l^*} + a_{i^*l^*+1}\tilde{q}_{i^*l^*+1} = 0$, leading to $\tilde{q}_{i^*l^*+1} = u+\frac{1}{u}+\frac{1}{a_{i^*l^*+1}}(W_{i^*j^*} - a_{i^*j^*}) - (1-\frac{2}{u})\frac{a_{i^*l^*}}{a_{i^*l^*+1}}$. If $\Delta_{l^*+1} < 1-\frac{2}{u}$, we stop. Otherwise, try the same way $\Delta_{l^*+2} = \frac{1}{a_{i^*l^*+2}}(W_{i^*j^*} - a_{i^*j^*}) - (1-\frac{2}{u})\frac{a_{i^*l^*}+a_{i^*l^*+1}}{a_{i^*l^*+2}}$, which would lead to $-(u+\frac{1}{u})a_{i^*l^*+2} - (\tilde{W}_{i^*j^*} - a_{i^*j^*}) + (1-\frac{2}{u})(a_{i^*l^*}+a_{i^*l^*+1}) + a_{i^*l^*+2}\tilde{q}_{i^*l^*+2} = 0$ and so on until getting $\Delta < 1-\frac{2}{u}$. Successful termination of this greedy procedure is guaranteed by condition (b): in the case of all $\Delta = 1-\frac{2}{u}$, i.e. $\tilde{q}_{i^*l} = u+1-\frac{1}{u}\ \forall l \neq j^*$, (19) would give $a_{i^*j^*} \leq 1-(1-\frac{1}{u})\sum_{\gamma\in J^+_{i^*}-\{j^*\}} a_{i^*\gamma}$ (contradicting (b)).

In the considered case, the inequality becomes $A_{i^*j^*} + \sum_{l\in J^+_{i^*}-\{j^*\}} \omega_{i^*l}\tilde{q}_{i^*l} \leq u+1-\frac{1}{u}$ after plugging y-values. Consider y-subvector with $1 < K \leq J-1$ zero values $y_{i^*l} = 0$ (including $y_{i^*j^*} = 0$). Now, the inequality is $A_{i^*j^*} - B_{i^*j^*}(K-1) + \sum_{l\in J^+_{i^*}-\{j^*\}} \omega_{i^*l}\tilde{q}_{i^*l} \leq u+1-\frac{1}{u}$ and its validness easily follows from condition (c).

Second, consider the q-subvector related to some $y_{i^*l^*} = 0$, $l^* \neq j^*$ and other $y_{i^*l} = 1$, $l \neq l^*$. It follows: $\tilde{q}_{i^*l^*} = u+1$. We take $\tilde{q}_{i^*j^*} = u+\frac{1}{u}$ and use a similar greedy approach starting with $\Delta_{l^*+1} = \frac{1}{a_{i^*l^*+1}}(W_{i^*l^*} - a_{i^*l^*})$. Using $B_{i^*j^*}(J-2) + C_{i^*j^*} = \frac{1}{u} - B_{i^*j^*}$, we have $(\frac{1}{u}+1-\frac{2}{u}-\frac{\tilde{W}_{i^*j^*}-\frac{1}{u}a_{i^*j^*}}{\sum_{\gamma\in J^+_{i^*}-\{j^*\}} a_{i^*\gamma}} - u - 1)\sum_{\gamma\in J^+_{i^*}-\{j^*\}} a_{i^*\gamma} + \sum_{\gamma\in J^+_{i^*}-\{j^*,l^*\}} a_{i^*\gamma}\tilde{q}_{i^*\gamma} + (u+1)a_{i^*l^*} = -(u+\frac{1}{u})\sum_{\gamma\in J^+_{i^*}-\{j^*\}} a_{i^*\gamma} - (\tilde{W}_{i^*j^*} - \frac{1}{u}a_{i^*j^*}) + (u+\frac{1}{u})\sum_{\gamma\in J^+_{i^*}-\{j^*,l^*,l^*+1\}} a_{i^*\gamma} + a_{i^*l^*+1}\tilde{q}_{i^*l^*+1} + (u+1)a_{i^*l^*} = -(u+\frac{1}{u})a_{i^*l^*+1} - \frac{1}{u}a_{i^*l^*} - \tilde{W}_{i^*j^*} + \frac{1}{u}a_{i^*j^*} + a_{i^*l^*+1}\tilde{q}_{i^*l^*+1} + a_{i^*l^*} = -(u+\frac{1}{u})a_{i^*l^*+1} - \tilde{W}_{i^*l^*} + a_{i^*l^*+1}\tilde{q}_{i^*l^*+1} + a_{i^*l^*} = 0$, confirming $\tilde{q}_{i^*l^*+1} < u+1-\frac{1}{u}$. Otherwise, continue with Δ_{l^*+2}. Because $\tilde{q}_{i^*j^*} = u+\frac{1}{u}$ is predetermined, two main outcome differences with the first case include the possibility to have all $\Delta = 1-\frac{2}{u}$ and the pathological variant of "not starting", i.e. all $\Delta = 0$. We admit both exceptions. What is important for us is the requirement for all $l \neq j^*$ to have the opportunity to be in the role of l^*, and this is guaranteed by condition (a).

In the considered case, the inequality becomes $-B_{i^*j^*} + \sum_{l \in J^+_{i^*} - \{j^*\}} \omega_{i^*l}\tilde{q}_{i^*l} \leq u + 1 - \frac{1}{u}$ after plugging y-values. Consider y-subvector with $1 < K \leq J-1$ zero values $y_{i^*l} = 0$ (excluding $y_{i^*j^*}$, which is fixed to 1). Now, the inequality is $-B_{i^*j^*}K + \sum_{l \in J^+_{i^*} - \{j^*\}} \omega_{i^*l}\tilde{q}_{i^*l} \leq u + 1 - \frac{1}{u}$ and its validness also easily follows.

Set 2. (J vectors).

Like in the previous theorems, this vector set is an abridged version of Set 1 with the propagation of $\tilde{q} = u + 1$ related to $t = 0$.

Set 3. (JI vectors).

Here we firstly can indicate two vectors in which all binary values =1 and all $\tilde{q}_{i^*l}$ $\forall l \neq j^*$ have values $= u + 1 - \frac{1}{u}$ with $\tilde{q}_{i^*j^*} = u + \frac{1}{u}$ and $\tilde{q}_{i^*j^*} = min(u + 1 - \frac{1}{u}, u + \frac{1}{a_{i^*j^*}}(1 - (1 - \frac{1}{u})\sum_{l \in J^+_i - \{j\}} a_{i^*l}))$, respectively. Indeed, applying again $B_{i^*j^*}(J-1) + C_{i^*j^*} = \frac{1}{u}$, we have $\sum_{l \in J^+_{i^*} - \{j\}} \omega_{i^*l}\tilde{q}_{i^*l} \leq u + 1 - \frac{1}{u}$ and for equality we take all $\tilde{q}_{i^*l} = u + 1 - \frac{1}{u}$ $\forall l \neq j^*$. Plus, the choice of one $\tilde{q}_{i^*j^*} = u + \frac{1}{u}$ is natural and one more $u + \frac{1}{u} < \tilde{q}_{i^*j^*} \leq u + 1 - \frac{1}{u}$ can be defined from plugging all $\tilde{q}_{i^*l} = u + 1 - \frac{1}{u}$ $\forall l \neq j^*$ to (10), which gives $\tilde{q}_{i^*j^*} \leq u + \frac{1}{a_{i^*j^*}}(1 - (1 - \frac{1}{u})\sum_{l \in J^+_i - \{j\}} a_{i^*l})$. In addition, the inequality $u + \frac{1}{u} < u + \frac{1}{a_{i^*j^*}}(1 - (1 - \frac{1}{u})\sum_{l \in J^+_i - \{j\}} a_{i^*l})$ is precisely condition (c).

To get $J-2$ vectors more, we return to Set 1, part 1. Try without loss of generality $l^* + 1 \neq j^*$ in $J^+_{i^*}$ with $\tilde{q}_{i^*l^*+1} = u + \frac{1}{u} + \frac{1}{a_{i^*l^*+1}}(\tilde{W}_{i^*j^*} - a_{i^*j^*})$ and the remaining $\tilde{q}_{i^*l} = u + \frac{1}{u}$, $l \neq \{j^*, l^* + 1\}$ for the same greedy procedure described above. After that take $l^* + 2 \neq j^*$ and so on. The process can successfully start from any $l \neq \{j^*, l^*\}$ (due to condition (d), in particular) and terminate. $\square$

One illustrative example for $I = 2,\ J = 3,\ i^* = 1,\ j^* = 1$ is in Table 10, where, according to the description in the proof, $\hat{Y}_{12} = u + \frac{1}{u} + min(1 - \frac{2}{u}, \frac{1}{a_{12}}(\tilde{W}_{11} - a_{11}))$, $\hat{Y}_{32} = u + \frac{1}{u} + min(1 - \frac{2}{u}, \frac{1}{a_{12}}(\tilde{W}_{13} - a_{13}))$, $\hat{Y}_{23} = u + \frac{1}{u} + min(1 - \frac{2}{u}, \frac{1}{a_{13}}(\tilde{W}_{12} - a_{12}))$, $R_1 = min(u + 1 - \frac{1}{u}, u + \frac{1}{a_{11}}(1 - (1 - \frac{1}{u})(a_{12} + a_{13}))$, $\hat{Y}_{13} = u + \frac{1}{u} + min(1 - \frac{2}{u}, \frac{1}{a_{13}}(\tilde{W}_{11} - a_{11}))$; (...) is related to the respective expression with $W - a$ (including possible $u + 1 - \frac{1}{u}$)

or $u + \frac{1}{u}$.

Table 10: Affinely independent vectors in Theorem 3.3.3

t_1	t_2	t_3	y_{11}	y_{12}	y_{13}	y_{21}	y_{22}	y_{23}	$\tilde{q}_{11}$	$\tilde{q}_{12}$	$\tilde{q}_{13}$	$\tilde{q}_{21}$	$\tilde{q}_{22}$	$\tilde{q}_{23}$
1	1	1	0	1	1	1	1	1	u+1	$\hat{Y}_{12}$	$(\ldots)$	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$
1	1	1	1	0	1	1	1	1	$u+\frac{1}{u}$	u+1	$\hat{Y}_{23}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$
1	1	1	1	1	0	1	1	1	$u+\frac{1}{u}$	$\hat{Y}_{32}$	u+1	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$
1	1	1	1	1	1	0	1	1	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$	u+1	$u+\frac{1}{u}$	$u+\frac{1}{u}$
1	1	1	1	1	1	1	0	1	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$	u+1	$u+\frac{1}{u}$
1	1	1	1	1	1	1	1	0	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$	u+1
0	1	1	0	1	1	0	1	1	u+1	$\hat{Y}_{12}$	$(\ldots)$	u+1	$u+\frac{1}{u}$	$u+\frac{1}{u}$
1	0	1	1	0	1	1	0	1	$u+\frac{1}{u}$	u+1	$\hat{Y}_{23}$	$u+\frac{1}{u}$	u+1	$u+\frac{1}{u}$
1	1	0	1	1	0	1	1	0	$u+\frac{1}{u}$	$\hat{Y}_{32}$	u+1	$u+\frac{1}{u}$	$u+\frac{1}{u}$	u+1
1	1	1	1	1	1	1	1	1	$u+\frac{1}{u}$	$u+1-\frac{1}{u}$	$u+1-\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$
1	1	1	1	1	1	1	1	1	R_1	$u+1-\frac{1}{u}$	$u+1-\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$
1	1	1	0	1	1	1	1	1	$u+1$	$(\ldots)$	$\hat{Y}_{13}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$
1	1	1	1	1	1	1	1	1	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+1-\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$
1	1	1	1	1	1	1	1	1	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+1-\frac{1}{u}$	$u+\frac{1}{u}$
1	1	1	1	1	1	1	1	1	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+1-\frac{1}{u}$

Example 3.3.2. $I = 1,\ J = 5,\ u = 5,\ a_{11} = \frac{21}{50},\ a_{12} = \frac{3}{5},\ a_{13} = \frac{9}{20},\ a_{14} = \frac{1}{2},\ a_{15} = \frac{12}{25}$.

See Figure 7 for a fragment of the conflict graph (without $\bar{\tilde{q}}$ variables).

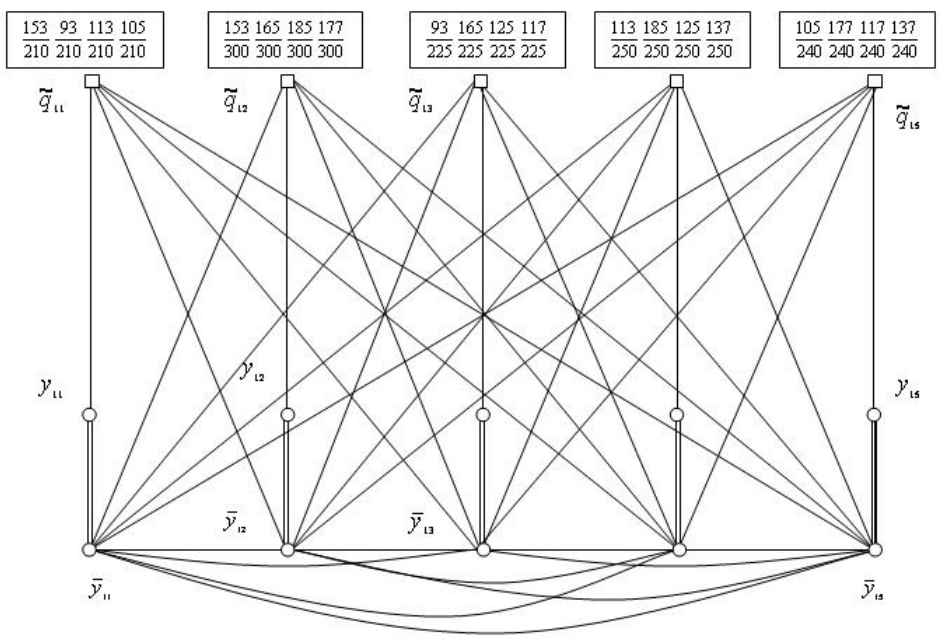

Figure 7: A fragment of the conflict graph with an "easy" structure

We call this structure "easy" because all variables $\bar{y}$ create the clique, plus, all weights of the mixed edges between variables $\tilde{q}$ and $\bar{y}$ are greater than $\frac{1}{u}$, i.e. $> \frac{1}{5}$.

Five mixed clique and five mixed star-clique are the only nontrivial facet-defining inequalities in this example. We do not have any weighted complementary inequality because condition (c) of Theorem 3.3.3 is not satisfied. To motivate new results let us modify this example.

Example 3.3.3. $I = 1,\ J = 5,\ u = 5,\ a_{11} = \frac{1}{4},\ a_{12} = \frac{3}{5},\ a_{13} = \frac{2}{5},\ a_{14} = \frac{1}{2},\ a_{15} = \frac{12}{25}.$

See Figure 8 for a fragment of the conflict graph (without $\bar{\tilde{q}}$ variables).

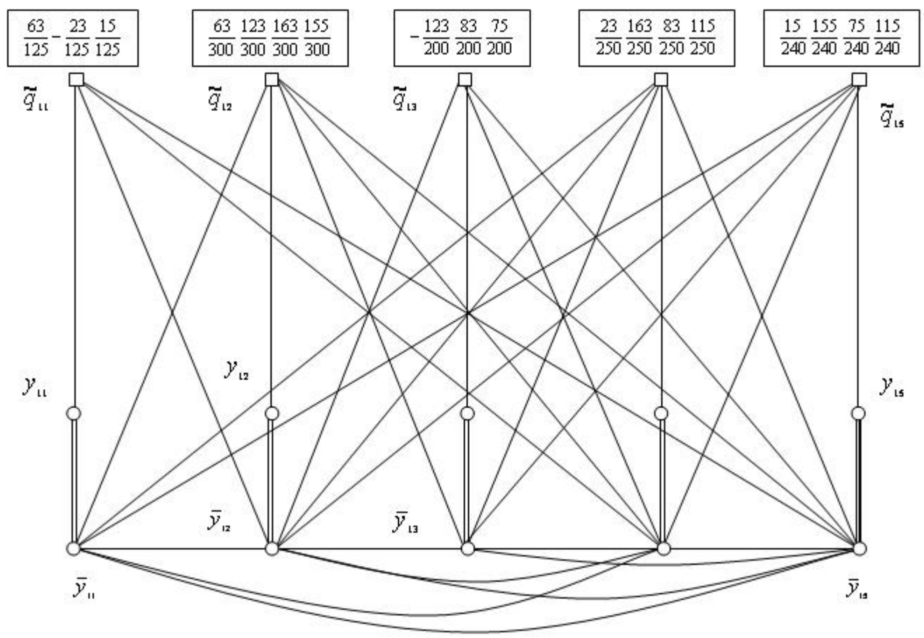

Figure 8: A fragment of the conflict graph for introduction of new inequalities

The deviations from the "easy structure" of the previous example are obvious: the absence of the binary edge between $\bar{y}_{11}$ and $\bar{y}_{13}$ (and respective mixed edges, or in other words, the weights of edges connecting $\tilde{q}_{11}$ with $\bar{y}_{13}$ and $\tilde{q}_{13}$ with $\bar{y}_{11}$ =0); plus weights of edges $\bar{y}_{11}$ - $\tilde{q}_{14}$ and $\bar{y}_{11}$ - $\tilde{q}_{15}$ are less than $\frac{1}{u}$. Everything remains all right for $\tilde{q}_{12}$, and this variable is in the center of the attention of the new result.

Theorem 3.3.4. *The following inequalities are facet-defining for conv(S^-)*

$$(1-\frac{\tilde{W}_{ij}}{a_{ij}})y_{ij}+\frac{1-\frac{1}{u}}{a_{ij}}\sum_{l\in J_i^+-\{j\}}a_{il}(1-y_{il})+\tilde{q}_{ij}\leq\ u+1$$

for $i,j\in J_i^+$ satisfying all the following conditions:

a) $\frac{1}{a_{ij}}(\tilde{W}_{il}-a_{il})\leq 1-\frac{2}{u}\ \forall l\neq j\in J_i^+$*;*

b) $\tilde{W}_{ij}-\frac{1}{u}a_{ij}\geq(1-\frac{1}{u})(a_{ir_1}+a_{ir_2})$*, where* a_{ir_1} *and* a_{ir_2} *are two smallest values among* $a_{il},\ l\in J_i^+-\{j\}$*;*

c) $\tilde{W}_{ij}>a_{ij}$

Proof. First, for validity, like in Theorem 3.3.2, we cannot have simultaneously $y_{il}=0$, y_{ij}=0 and the conflict graph edge weight $1-\frac{1}{u}-\frac{1}{a_{ij}}(\tilde{W}_{il}-a_{il})>0,\ l\neq j$ (y_{ij} would be 1) due to condition (a), which is even stronger: $1-\frac{1}{u}-\frac{1}{a_{ij}}(\tilde{W}_{il}-a_{il})\geq\frac{1}{u}$. Also, we can observe that subtraction the coefficient of y_{ij} from this positive weight would give precisely the coefficient of $1-y_{il}$, i.e. $1-\frac{1}{u}-\frac{1}{a_{ij}}(\tilde{W}_{il}-a_{il})-(1-\frac{\tilde{W}_{ij}}{a_{ij}})=\frac{\tilde{W}_{ij}-\tilde{W}_{il}+a_{il}-\frac{1}{u}a_{ij}}{a_{ij}}=\frac{\frac{1}{u}(a_{ij}-a_{il})+a_{il}-\frac{1}{u}a_{ij}}{a_{ij}}=\frac{a_{il}(1-\frac{1}{u})}{a_{ij}}$. Condition (b) says about necessity of having at least one pair of u+1 values. Suppose that we have $y_{il_1}=y_{il_2}=0$ (with $\tilde{q}_{il_1}=\tilde{q}_{il_2}=0$), $l_1\neq j\neq l_2$; plugging these values to (19) provides $\tilde{q}_{ij}\leq u+\frac{1}{a_{ij}}(1-a_{il_1}-a_{il_2}-\frac{1}{u}\sum_{l\in J_i^+-\{l_1,j,l_2\}}a_{il})=u+\frac{1}{a_{ij}}(\tilde{W}_{ij}-(1-\frac{1}{u})(a_{il_1}+a_{il_2}))$. Together with the requirement $\tilde{q}_{ij}\geq 1+\frac{1}{u}$ it gives $\tilde{W}_{ij}-\frac{1}{u}a_{ij}\geq(1-\frac{1}{u})(a_{il_1}+a_{il_2})$. In addition, we have the following $J+2JI$ affinely independent points (where i* or j* denotes one index in consideration respectively among i and j), satisfying the inequality at equality:

1) JI vectors like in set 1 of Theorem 3.3.2, but $\tilde{q}_{i^*j^*}=u+\frac{1}{u}+\frac{1}{a_{i^*j^*}}(\tilde{W}_{i^*l}-a_{i^*l})$, (related to $y_{i^*l}=0,\ l\neq j^*$) because of condition (a);

2) J vectors like in set 2 of Theorem 3.3.2 with the same exception as in set 1: $\tilde{q}_{i^*j^*}=u+\frac{1}{u}+\frac{1}{a_{i^*j^*}}(\tilde{W}_{i^*l}-a_{i^*l})$(related to $t_j=0,\ j\neq j^*$);

3) JI vectors, in which WLOG first J have to be different from those in Theorem 3.3.2 because we cannot have "all binary variables = 1" situation here. One vector is

special and follows from condition (b): $y_{i^*r_1} = y_{i^*r_2} = 0$, all other binary variables=1; $\tilde{q}_{i^*j^*} = u + \frac{1}{a_{i^*j^*}}(\tilde{W}_{i^*j^*} - (1 - \frac{1}{u})(a_{i^*r_1} + a_{i^*r_2})), \tilde{q}_{i^*r_1} = \tilde{q}_{i^*r_2} = u + 1$, other $\tilde{q} = u + \frac{1}{u}$. Plus, $J - 1$ vectors have $y_{i^*j^*} = 0$, all other binary variables=1; $\tilde{q}_{i^*j^*} = u + 1$, one $\tilde{q}_{i^*l} = u + \frac{1}{u} + min(1 - \frac{2}{u}, \frac{1}{a_{i^*l^*}}(\tilde{W}_{i^*j^*} - a_{i^*j^*}))$, $l \neq j^*$, other $\tilde{q} = u + \frac{1}{u}$. Condition (c) supports having distinct vectors: each $l \neq j^*$ can participate. $\square$

One illustrative example for $I = 2,\ J = 3,\ i^* = 1,\ j^* = 1$ is in Table 11, where $Y_{12} = u + \frac{1}{u} + \frac{1}{a_{11}}(\tilde{W}_{12} - a_{12})$, $Y_{13} = u + \frac{1}{u} + \frac{1}{a_{11}}(\tilde{W}_{13} - a_{13})$, $R_{1-23} = u + \frac{1}{a_{11}}(\tilde{W}_{11} - (1 - \frac{1}{u})(a_{12} + a_{13})$, $Y_{21} = u + \frac{1}{u} + min(1 - \frac{2}{u}, \frac{1}{a_{12}}(\tilde{W}_{11} - a_{11}))$, $Y_{31} = u + \frac{1}{u} + min(1 - \frac{2}{u}, \frac{1}{a_{13}}(\tilde{W}_{11} - a_{11}))$.

Table 11: Affinely independent vectors in Theorem 3.3.4

t_1	t_2	t_3	y_{11}	y_{12}	y_{13}	y_{21}	y_{22}	y_{23}	$\tilde{q}_{11}$	$\tilde{q}_{12}$	$\tilde{q}_{13}$	$\tilde{q}_{21}$	$\tilde{q}_{22}$	$\tilde{q}_{23}$
1	1	1	0	1	1	1	1	1	u+1	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$
1	1	1	1	0	1	1	1	1	Y_{12}	u+1	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$
1	1	1	1	1	0	1	1	1	Y_{13}	$u+\frac{1}{u}$	u+1	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$
1	1	1	1	1	1	0	1	1	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$	u+1	$u+\frac{1}{u}$	$u+\frac{1}{u}$
1	1	1	1	1	1	1	0	1	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$	u+1	$u+\frac{1}{u}$
1	1	1	1	1	1	1	1	0	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$	u+1
0	1	1	0	1	1	0	1	1	u+1	$u+\frac{1}{u}$	$u+\frac{1}{u}$	u+1	$u+\frac{1}{u}$	$u+\frac{1}{u}$
1	0	1	1	0	1	1	0	1	Y_{12}	u+1	$u+\frac{1}{u}$	$u+\frac{1}{u}$	u+1	$u+\frac{1}{u}$
1	1	0	1	1	0	1	1	0	Y_{13}	$u+\frac{1}{u}$	u+1	$u+\frac{1}{u}$	$u+\frac{1}{u}$	u+1
1	1	1	1	0	0	1	1	1	R_{1-23}	$u+1$	$u+1$	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$
1	1	1	0	1	1	1	1	1	$u+1$	Y_{21}	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$
1	1	1	0	1	1	1	1	1	$u+1$	$u+\frac{1}{u}$	Y_{31}	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$
1	1	1	1	1	1	1	1	1	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+1-\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$
1	1	1	1	1	1	1	1	1	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+1-\frac{1}{u}$	$u+\frac{1}{u}$
1	1	1	1	1	1	1	1	1	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+\frac{1}{u}$	$u+1-\frac{1}{u}$

Example 3.3.1 (continued). Inequality (d) of this example, $-63y_{11} - 19y_{12} - 60y_{13} + 140\tilde{q}_{12} \leq 577$ comes with satisfying all conditions of Theorem 3.3.4. It also can be written in the form $\frac{63}{140}(1 - y_{11}) - \frac{19}{140}y_{12} + \frac{60}{140}(1 - y_{13}) + \tilde{q}_{12} \leq 5$.

Example 3.3.3 (continued). All conditions of Theorem 3.3.4 are satisfied for a_{12}. So, we have the facet-defining inequality $\frac{1}{3}(1 - y_{11}) - \frac{37}{300}y_{12} + \frac{8}{15}(1 - y_{13}) + \frac{2}{3}(1 - y_{14}) + \frac{16}{25}(1 - y_{15}) + \tilde{q}_{12} \leq 6$. We can observe the construction of the affinely independent points in Table 12 (recall that $j^* = 2$ and we have the respective "shift" comparing to the previous table).

The condition (a) in Theorem 3.3.4 is conservative and sometimes we can have the same facet-defining inequalities without satisfying it.

Table 12: Affinely independent vectors in Example 3.3.3

t_1	t_2	t_3	t_4	t_5	y_{11}	y_{12}	y_{13}	y_{14}	y_{15}	$\tilde{q}_{11}$	$\tilde{q}_{12}$	$\tilde{q}_{13}$	$\tilde{q}_{14}$	$\tilde{q}_{15}$
1	1	1	1	1	1	0	1	1	1	26/5	6	26/5	26/5	26/5
1	1	1	1	1	1	1	0	1	1	26/5	559/100	6	26/5	26/5
1	1	1	1	1	1	1	1	0	1	26/5	1637/300	26/5	6	26/5
1	1	1	1	1	1	1	1	1	0	26/5	329/60	26/5	26/5	6
1	1	1	1	1	0	1	1	1	1	6	579/100	26/5	26/5	26/5
1	0	1	1	1	1	0	1	1	1	26/5	6	26/5	26/5	26/5
1	1	0	1	1	1	1	0	1	1	26/5	559/100	6	26/5	26/5
1	1	1	0	1	1	1	1	0	1	26/5	1637/300	26/5	6	26/5
1	1	1	1	0	1	1	1	1	0	26/5	329/60	26/5	26/5	6
0	1	1	1	1	0	1	1	1	1	6	579/100	26/5	26/5	26/5
1	1	1	1	1	0	1	0	1	1	6	1577/300	6	26/5	26/5
1	1	1	1	1	1	0	1	1	1	26/5	6	1077/200	26/5	26/5
1	1	1	1	1	1	0	1	1	1	26/5	6	26/5	1337/250	26/5
1	1	1	1	1	1	0	1	1	1	26/5	6	26/5	26/5	257/48
1	1	1	1	1	1	0	1	1	1	687/125	6	26/5	26/5	26/5

Theorem 3.3.5. *The facet-defining inequalities of Theorem 3.3.4 remain actual if we substitute conditions (a) and (c) by the following:*

a') $\exists l^* \neq \{j, r_1, r_2\}$, where r_1, r_2 are from Theorem 3.3.4, such that $\frac{1}{a_{ij}}(\tilde{W}_{il^*} - a_{il^*}) \leq 1 - \frac{2}{u}$;

c') $0 \leq \frac{1}{a_{ij}}(\tilde{W}_{il^*} - a_{il^*} - a_{il}(1 - \frac{1}{u})) \leq \frac{2}{u}$ $\forall l \neq \{j,\ l^*\} \in J_i^+$.

Proof. Comparing to Theorem 3.3.4, we have transferring a part of the value of $\tilde{q}_{ij}$ to the jump of $\tilde{q}_{il}$ from $u + \frac{1}{u}$ to $u+1$ (and respective change of y_{il} from 1 to 0), keeping the balance. Consider $\tilde{q}_{il^*} = u+1, \tilde{q}_{ij} = u + \frac{1}{u} + \Delta_j, \tilde{q}_{il} = u + \frac{1}{u} + \Delta_l$ with (in contrast to Theorem 3.3.3) discrete $\Delta_l \in \{0, 1 - \frac{1}{u}\}$. Plugging these values (together with all other $\tilde{q} = u + \frac{1}{u}$) to (10), we have $\Delta_j a_{ij} + \Delta_l a_{il} \leq \tilde{W}_{il^*} - a_{il^*}$. Existence of $\Delta_l = 1 - \frac{1}{u}$ means $\Delta_j a_{ij} \leq \tilde{W}_{il^*} - a_{il^*} - a_{il}(1 - \frac{1}{u})$ and condition (c'). The value $\Delta_l = 0$ just means $\Delta_j \leq \frac{1}{a_{ij}}(\tilde{W}_{il} - a_{il}) \leq 1 - \frac{2}{u}$ (see Theorem 3.3.2). So, in the structure of the affinely independent points we keep the same "u+1" diagonal as in Theorem 3.3.4, but include jumps for all $l \neq \{j,\ l^*\}$. □

Example 3.3.4. $I = 1,\ J = 4,\ u = 5,\ a_{11} = \frac{1}{4},\ a_{12} = \frac{3}{5},\ a_{13} = \frac{3}{10},\ a_{14} = \frac{1}{2}.$

See Figure 9 for a fragment of the conflict graph (without $\bar{\tilde{q}}$ variables).

The conditions of Theorem 3.3.5 are satisfied for a_{12} (i.e. $j = 2$), $l^* = 4$. So, we have the facet-defining inequality $\frac{1}{3}(1 - y_{11}) - \frac{19}{60}y_{12} + \frac{2}{5}(1 - y_{13}) + \frac{2}{3}(1 - y_{14}) + \tilde{q}_{12} \leq 6$. It is useful to compare the construction of key affinely independent points here and in the previous example. The fragment is in Table 13.

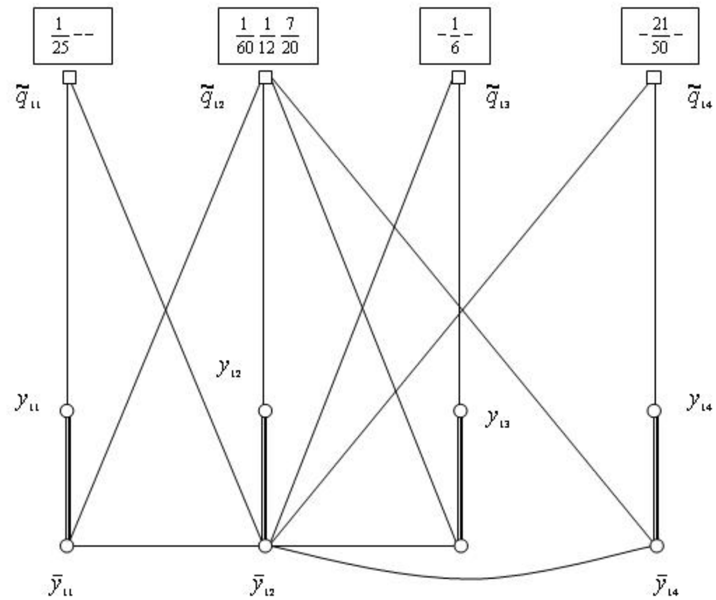

Figure 9: A fragment of the conflict graph for Example 3.3.4

Table 13: Some affinely independent vectors in Example 3.3.4

y_{11}	y_{12}	y_{13}	y_{14}	$\tilde{q}_{11}$	$\tilde{q}_{12}$	$\tilde{q}_{13}$	$\tilde{q}_{14}$
1	1	1	0	26/5	113/20	26/5	6
0	1	1	0	6	319/60	26/5	6
1	1	0	0	26/5	21/4	6	6
0	1	0	1	6	67/12	6	26/5

The very last row in Table 13 is similar to the first one in the third set of vectors in Theorem 3.3.4 and respective Example 3.3.3. But we no longer can have the situation of "only one $u+1$" in each row of the first two sets. Only one special variable ("base element") can have this situation (and this is the key difference between Theorems 3.3.4 and 3.3.5). Nevertheless, we can have the same inequality because we can construct such affinely independent points with a couple of $u+1$ with the condition that $\tilde{q}_{ij} < u + 1 - \frac{1}{u}$ (here all numbers in the respective column are $< \frac{29}{5}$). When it is impossible, we can have inequalities with the modifications of the coefficients.

Example 3.3.5. $I = 1,\ J = 4,\ u = 4,\ a_{11} = \frac{1}{4},\ a_{12} = \frac{3}{5},\ a_{13} = \frac{3}{10},\ a_{14} = \frac{1}{2}.$

See Figure 10 for a fragment of the conflict graph (without $\bar{\tilde{q}}$ variables).

The only difference comparing to Example 3.3.4 is $u = 4$ (instead of $u = 5$).

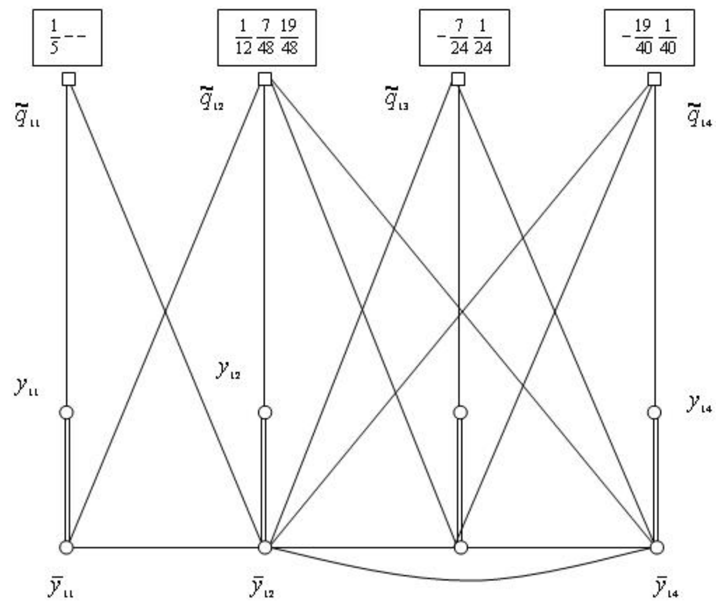

Figure 10: A fragment of the conflict graph for Example 5

Both Theorems 3.3.4 and 3.3.5 do not work here, but with some modification of the coefficients, we can construct the facet-defining inequality $\frac{3}{16}(1-y_{11}) - \frac{1}{16}y_{12} + \frac{5}{24}(1-y_{13}) + \frac{11}{24}(1-y_{14}) + \tilde{q}_{12} \leq 5$ with the following fragment of key affinely independent vectors (in Table 14).

Table 14: Some affinely independent vectors in Example 3.3.5

y_{11}	y_{12}	y_{13}	y_{14}	$\tilde{q}_{11}$	$\tilde{q}_{12}$	$\tilde{q}_{13}$	$\tilde{q}_{14}$
1	1	1	0	17/4	221/48	17/4	5
0	1	1	0	5	103/24	17/4	5
0	1	1	1	5	19/4	17/4	17/4
0	1	0	1	5	109/24	5	17/4

The coefficient of $(1-y_{11})$ is $\frac{a_{11}}{a_{12}}(1-\frac{1}{u})$ (as in Theorems 3.3.4 and 3.3.5), but for those of $(1-y_{13})$ and $(1-y_{14})$ we have respective modifications: $\frac{a_{13}}{a_{12}}(1-\frac{1}{u}) + 1 - \frac{2}{u} - \frac{\tilde{W}_{11}-a_{11}}{a_{12}}$ and $\frac{a_{14}}{a_{12}}(1-\frac{1}{u}) + 1 - \frac{2}{u} - \frac{\tilde{W}_{11}-a_{11}}{a_{12}}$. The coefficient of y_{12} is also modified as $1 - \frac{\tilde{W}_{12}}{a_{12}} - (1 - \frac{2}{u} - \frac{\tilde{W}_{11}-a_{11}}{a_{12}})$. The extra term $1 - \frac{2}{u} - \frac{\tilde{W}_{11}-a_{11}}{a_{12}}$ is brought by a_{11} and respective $\tilde{q}_{11}$ which created the $\frac{19}{4}$-situation ($u + 1 - \frac{1}{u}$ in general) in the table.

Example 3.3.3 (continued). Absolutely the same way, we can get modified facet-defining inequalities with $\tilde{q}_{14}$ and $\tilde{q}_{15}$. Recall that, in contrast to $\tilde{q}_{12}$, one positive weight (related to the mixed edges with $\bar{y}_1 1$) is less than $\frac{1}{u}$ for both $\tilde{q}_{14}$ and $\tilde{q}_{15}$. It makes the difference with $\tilde{q}_{12}$. We can derive the following facet-defining inequalities: $\frac{2}{5}(1-y_{11})+\frac{213}{250}(1-y_{12})+\frac{133}{250}(1-y_{13})-\frac{1}{5}y_{14}+\frac{33}{50}(1-y_{15})+\tilde{q}_{14} \leq 6$ and $\frac{5}{12}(1-y_{11})+\frac{69}{80}(1-y_{12})+\frac{127}{240}(1-y_{13})+\frac{167}{240}(1-y_{14})-\frac{16}{25}y_{15}+\tilde{q}_{15} \leq 6$. For former, we illustrate the details of key affinely independent vectors in Table 15.

Table 15: Some affinely independent vectors in Example 3.3.3

y_{11}	y_{12}	y_{13}	y_{14}	y_{15}	$\tilde{q}_{11}$	$\tilde{q}_{12}$	$\tilde{q}_{13}$	$\tilde{q}_{14}$	$\tilde{q}_{15}$
0	1	1	1	1	6	26/5	26/5	29/5	26/5
1	0	1	1	1	26/5	6	26/5	1337/250	26/5
1	1	0	1	1	26/5	26/5	6	1417/250	26/5
1	1	1	1	0	26/5	26/5	26/5	277/50	6
0	1	0	1	1	6	26/5	6	1317/250	26/5

The coefficient of $(1-y_{11})$ is $\frac{a_{11}}{a_{12}}(1-\frac{1}{u})$ (as in Theorems 3.3.4 and 3.3.5), but for those of $(1-y_{12})$, $(1-y_{13})$ and $(1-y_{15})$ we have respective modifications: $\frac{a_{12}}{a_{14}}(1-\frac{1}{u})+1-\frac{2}{u}-\frac{\tilde{W}_{11}-a_{11}}{a_{14}}$, $\frac{a_{13}}{a_{14}}(1-\frac{1}{u})+1-\frac{2}{u}-\frac{\tilde{W}_{11}-a_{11}}{a_{14}}$ and $\frac{a_{15}}{a_{14}}(1-\frac{1}{u})+1-\frac{2}{u}-\frac{\tilde{W}_{11}-a_{11}}{a_{14}}$. The coefficient of y_{14} is also modified as $1-\frac{\tilde{W}_{14}}{a_{14}}-(1-\frac{2}{u}-\frac{\tilde{W}_{11}-a_{11}}{a_{14}})$.

Example 3.3.6. $I=1,\ J=4,\ u=4,\ a_{11}=\frac{1}{4},\ a_{12}=\frac{3}{5},\ a_{13}=\frac{3}{10},\ a_{14}=\frac{2}{5}$.

See Figure 11 for a fragment of the conflict graph (without $\bar{\tilde{q}}$ variables).

The only difference comparing to Example 3.3.4 is the fact that the weight of the mixed edge connecting $\tilde{q}_{12}$ and $\bar{y}_{14}$ is no longer greater than $\frac{1}{u}$ (and, actually, none related to $\tilde{q}_{12}$). This example illustrates the general difficulty of such modifications in the form of the presence of multiple base elements. Consider three different facet-defining inequalities related to $\tilde{q}_{12}$, depending on how we use base elements:

1) $\frac{7}{24}(1-y_{11})-\frac{1}{24}y_{12}+\frac{1}{6}(1-y_{13})+\frac{7}{24}(1-y_{14})+\tilde{q}_{12} \leq 5$;

2) $\frac{1}{6}(1-y_{11})-\frac{5}{48}y_{12}+\frac{17}{48}(1-y_{13})+\frac{17}{48}(1-y_{14})+\tilde{q}_{12} \leq 5$;

3) $\frac{7}{24}(1-y_{11})-\frac{11}{48}y_{12}+\frac{17}{48}(1-y_{13})+\frac{23}{48}(1-y_{14})+\tilde{q}_{12} \leq 5$.

For the first inequality, see Table 16.

With the presence of two base elements, the coefficients of $(1-y_{11})$, y_{12}, $(1-y_{13})$ and $(1-y_{14})$ are respectively $\frac{a_{11}}{a_{12}}(1-\frac{1}{u})+1-\frac{2}{u}-\frac{\tilde{W}_{14}-a_{14}}{a_{12}}$, $1-\frac{\tilde{W}_{12}}{a_{12}}-(1-\frac{2}{u}-\frac{\tilde{W}_{11}-a_{11}}{a_{12}})-$

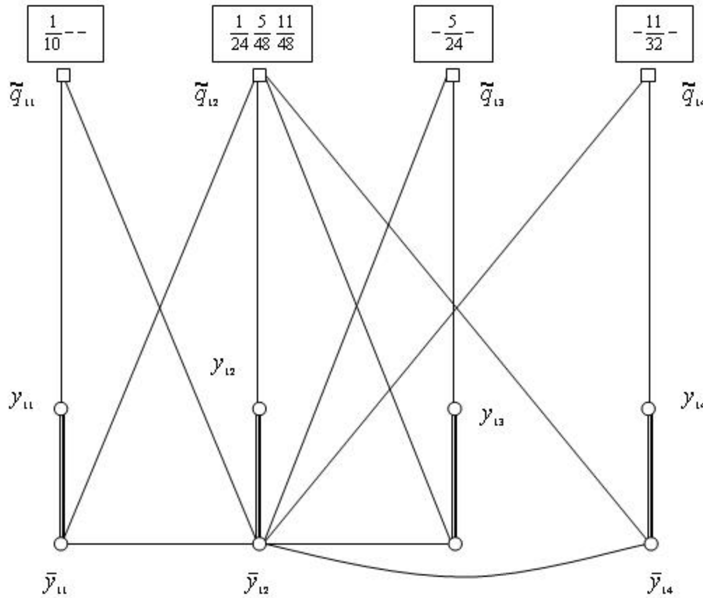

Figure 11: A fragment of the conflict graph for Example 3.3.6

Table 16: Affinely independent vectors in Example 3.3.6, inequality 1

y_{11}	y_{12}	y_{13}	y_{14}	$\tilde{q}_{11}$	$\tilde{q}_{12}$	$\tilde{q}_{13}$	$\tilde{q}_{14}$
1	1	1	0	17/4	19/4	17/4	5
0	1	1	0	5	107/24	17/4	5
0	1	1	1	5	19/4	17/4	17/4
0	1	0	1	5	55/12	5	17/4

$(1-\frac{2}{u}-\frac{\tilde{W}_{14}-a_{14}}{a_{12}})$, $\frac{a_{13}}{a_{12}}(1-\frac{1}{u})+1-\frac{2}{u}-\frac{\tilde{W}_{11}-a_{11}}{a_{12}}$, and $\frac{a_{14}}{a_{12}}(1-\frac{1}{u})+1-\frac{2}{u}-\frac{\tilde{W}_{11}-a_{11}}{a_{12}}$.

For the second inequality, see Table 17.

Table 17: Affinely independent vectors in Example 3.3.6, inequality 2

y_{11}	y_{12}	y_{13}	y_{14}	$\tilde{q}_{11}$	$\tilde{q}_{12}$	$\tilde{q}_{13}$	$\tilde{q}_{14}$
1	1	1	0	17/4	19/4	17/4	5
1	1	0	1	17/4	19/4	5	17/4
1	1	0	0	17/4	211/48	5	5
0	1	0	1	5	55/12	5	17/4

With the presence of two base elements, the coefficients of $(1-y_{11})$, y_{12}, $(1-y_{13})$ and $(1-y_{14})$ are respectively $\frac{a_{11}}{a_{12}}(1-\frac{1}{u})+1-\frac{2}{u}-\frac{\tilde{W}_{13}-a_{13}}{a_{12}}$, $1-\frac{\tilde{W}_{12}}{a_{12}}-(1-\frac{2}{u}-\frac{\tilde{W}_{13}-a_{13}}{a_{12}})-(1-\frac{2}{u}-\frac{\tilde{W}_{14}-a_{14}}{a_{12}})$, $\frac{a_{13}}{a_{12}}(1-\frac{1}{u})+1-\frac{2}{u}-\frac{\tilde{W}_{14}-a_{14}}{a_{12}}$, and $\frac{a_{14}}{a_{12}}(1-\frac{1}{u})+1-\frac{2}{u}-\frac{\tilde{W}_{13}-a_{13}}{a_{12}}$.

For the third inequality, see Table 18.

Table 18: Affinely independent vectors in Example 3.3.6, inequality 3

y_{11}	y_{12}	y_{13}	y_{14}	$\tilde{q}_{11}$	$\tilde{q}_{12}$	$\tilde{q}_{13}$	$\tilde{q}_{14}$
1	1	1	0	17/4	19/4	17/4	5
0	1	1	0	5	107/24	17/4	5
1	1	0	0	17/4	211/48	5	5
0	1	0	1	5	55/12	5	17/4

With the presence of the "double" base element, the coefficients of $(1-y_{11})$, y_{12}, $(1-y_{13})$ and $(1-y_{14})$ are respectively $\frac{a_{11}}{a_{12}}(1-\frac{1}{u})+1-\frac{2}{u}-\frac{\tilde{W}_{14}-a_{14}}{a_{12}}$, $1-\frac{\tilde{W}_{12}}{a_{12}}-2(1-\frac{2}{u}-\frac{\tilde{W}_{14}-a_{14}}{a_{12}})$, $\frac{a_{13}}{a_{12}}(1-\frac{1}{u})+1-\frac{2}{u}-\frac{\tilde{W}_{14}-a_{14}}{a_{12}}$, and $\frac{a_{14}}{a_{12}}(1-\frac{1}{u})+1-\frac{2}{u}-\frac{\tilde{W}_{14}-a_{14}}{a_{12}}$.

3.4 Separation and Computational Issues.

In this section, we apply the same separation scheme as in Chapter II.

Consider a maximal cut violation as a separation problem for the weighted complementary inequalities. Taking into account that $C_{ij}=\frac{1}{u}-B_{ij}(J-1)$ and after multiplying all terms by $\sum_{\gamma\in J_i^+-\{j\}}a_{i\gamma}$, they can be written in the form:

$$(1-y_{ij})(a_{ij}-\tilde{W}_{ij}+(1-\frac{2}{u})\sum_{\gamma\in J_i^+-\{j\}}a_{i\gamma})+(\sum_{l\in J_i^+-\{j\}}y_{il}-J+1)(-\frac{1}{u}a_{ij}+\tilde{W}_{ij}-(1-\frac{2}{u})$$

$$\sum_{\gamma\in J_i^+-\{j\}}a_{i\gamma})+\sum_{l\in J_i^+-\{j\}}a_{il}\tilde{q}_{il}\leq(u+1-\frac{1}{u})\sum_{\gamma\in J_i^+-\{j\}}a_{i\gamma}$$

Dropping for brevity index i, denote (different for each i) $\tilde{\Psi}=1-(1-\frac{1}{u})\sum_{\gamma\in J^+}a_\gamma$. Introducing the characteristic vector z (the unit vector for indicating variable j) for yet to be determined partition (j, all other $l\neq j$), we show that the separation problem for weighted complementary inequalities is polynomially solvable. Indeed, having an LP solution with $(y^*,\tilde{q}^*)$, the y_j-term $(1-y_j)(\frac{1}{u}a_j-\tilde{\Psi})$ becomes $\sum_\gamma(1-y^*_\gamma)(\frac{1}{u}a_\gamma-\tilde{\Psi})z_\gamma$; the $\tilde{q}$-term is $\sum_\gamma a_\gamma\tilde{q}^*_\gamma(1-z_\gamma)$; the right hand side is $(u+1-\frac{1}{u})\sum_\gamma a_\gamma(1-z_\gamma)$; the y_l-term $(\sum y_l-J+1)(\tilde{\Psi}+(1-\frac{2}{u})a_j)$ can be viewed initially as quadratic with respect to z: $(\sum_\gamma y^*_\gamma(1-z_\gamma)-J+1)(\tilde{\Psi}+(1-\frac{2}{u})\sum_\gamma a_\gamma z_\gamma)$, but it can be simplified into a linear one combining with y_j-term and using the facts that $z_{\gamma_1}z_{\gamma_2}=0$, $\sum_\gamma z_\gamma=1$, and $z^2_\gamma=z_\gamma$ in the following way:

$(\tilde{\Psi}+(1-\frac{2}{u})\sum_\gamma a_\gamma z_\gamma)(\sum_\gamma y^*_\gamma(1-z_\gamma)-J+1)+\sum_\gamma(1-y^*_\gamma)(\frac{1}{u}a_\gamma-\tilde{\Psi})z_\gamma=\tilde{\Psi}\sum_\gamma y^*_\gamma-$

$\tilde{\Psi}\sum_\gamma y^*_\gamma z_\gamma + \tilde{\Psi}(1-J) + (1-\frac{2}{u})\sum_\gamma a_\gamma z_\gamma \sum_\gamma y^*_\gamma - (1-\frac{2}{u})\sum_\gamma a_\gamma z_\gamma \sum_\gamma y^*_\gamma z_\gamma + (1-\frac{2}{u})\sum_\gamma a_\gamma z_\gamma(1-J) + \frac{1}{u}\sum_\gamma a_\gamma z_\gamma - \tilde{\Psi}\sum_\gamma z_\gamma - \frac{1}{u}\sum_\gamma y^*_\gamma a_\gamma z_\gamma + \tilde{\Psi}\sum_\gamma y^*_\gamma z_\gamma = (\tilde{\Psi} + (1-\frac{2}{u})\sum_\gamma a_\gamma z_\gamma)\sum_\gamma y^*_\gamma - \tilde{\Psi}J - (1-\frac{2}{u})\sum_\gamma a_\gamma z_\gamma y^*_\gamma + (1-\frac{2}{u})\sum_\gamma a_\gamma z_\gamma(1-J) + \frac{1}{u}\sum_\gamma a_\gamma z_\gamma - \frac{1}{u}\sum_\gamma y^*_\gamma a_\gamma z_\gamma = (\tilde{\Psi} + (1-\frac{2}{u})\sum_\gamma a_\gamma z_\gamma)\sum_\gamma y^*_\gamma - \tilde{\Psi}J - (1-\frac{1}{u})\sum_\gamma a_\gamma z_\gamma y^*_\gamma + (1-\frac{1}{u}+(\frac{2}{u}-1)J)\sum_\gamma a_\gamma z_\gamma$.

The separation objective function (subject to the unit vector z) becomes $max_z(\sum_\gamma y^*_\gamma)(\tilde{\Psi} + (1-\frac{2}{u})\sum_\gamma a_\gamma z_\gamma) - \tilde{\Psi}J - (1-\frac{1}{u})\sum_\gamma a_\gamma z_\gamma y^*_\gamma + (1-\frac{1}{u}+(\frac{2}{u}-1)J)\sum_\gamma a_\gamma z_\gamma + \sum_\gamma a_\gamma \tilde{q}^*_\gamma(1-z_\gamma) - (u+1-\frac{1}{u})\sum_\gamma a_\gamma(1-z_\gamma)$. The z-coefficient with the maximal value gives j in the partition with the condition that an optimal objective value is positive. So, we are looking for index j^*, if any, which is related to the maximal value ($j^* = argmax_j$) among $a_j(\sum_\gamma y^*_\gamma(1-\frac{2}{u}) - (1-\frac{1}{u})y^*_j - \tilde{q}^*_j + (\frac{2}{u}-1)(J-1)+u+1)$ with the condition that this value $> \sum_\gamma(a_\gamma(u+1-\frac{1}{u}-\tilde{q}^*_\gamma) - \tilde{\Psi}y^*_\gamma) + \tilde{\Psi}J$.

Similarly, the mixed star-clique inequalities and incomplete linking inequalities are polynomially separable. After weakening, multiplying each term by a_{ij} and dropping for brevity index i, denote (different for each i) $\tilde{\Phi} = 1 - \frac{1}{u}\sum_{\gamma\in J^+} a_\gamma$. This number is connected with the previous notations as follows: $\tilde{\Phi} = W_\gamma - \frac{1}{u}a_\gamma$.

The mixed star-clique inequalities can be written in the form:

$$\frac{1}{u}a_j y_j + \sum_{l\neq j}((1-\frac{2}{u})a_j - \tilde{\Phi} + (1-\frac{1}{u})a_l)(1-y_l) + a_j\tilde{q}_j \leq (u+1)a_j$$

After introducing the characteristic vector z, the y_j-term becomes $\frac{1}{u}\sum_\gamma a_\gamma y^*_\gamma z_\gamma$; the $\tilde{q}$-term is $\sum_\gamma a_\gamma \tilde{q}^*_\gamma z_\gamma$; the right hand side is $(u+1)\sum_\gamma a_\gamma z_\gamma$; the y_l-term can be viewed initially as quadratic with respect to z: $(-\tilde{\Phi} + (1-\frac{2}{u})\sum_\gamma a_\gamma z_\gamma)\sum_\gamma(1-y^*_\gamma)(1-z_\gamma) + (1-\frac{1}{u})\sum_\gamma a_\gamma(1-y^*_\gamma)(1-z_\gamma)$, but it can be simplified into a linear one combining with y_j-term and using the facts that $z_{\gamma_1}z_{\gamma_2} = 0$, $\sum_\gamma z_\gamma = 1$, and $z^2_\gamma = z_\gamma$ in the following way:

$(-\tilde{\Phi} + (1-\frac{2}{u})\sum_\gamma a_\gamma z_\gamma)\sum_\gamma(1-y^*_\gamma) + \tilde{\Phi}\sum_\gamma(1-y^*_\gamma)z_\gamma - (1-\frac{2}{u})\sum_\gamma a_\gamma z_\gamma \sum_\gamma(1-y^*_\gamma)z_\gamma + (1-\frac{1}{u})\sum_\gamma a_\gamma(1-y^*_\gamma) - (1-\frac{1}{u})\sum_\gamma a_\gamma(1-y^*_\gamma)z_\gamma + \frac{1}{u}\sum_\gamma a_\gamma y^*_\gamma z_\gamma = (-\tilde{\Phi} +$

$(1-\frac{2}{u})\sum_\gamma a_\gamma z_\gamma)\sum_\gamma(1-y^*_\gamma) + \tilde{\Phi}\sum_\gamma z_\gamma - \tilde{\Phi}\sum_\gamma y^*_\gamma z_\gamma - (1-\frac{2}{u})\sum_\gamma a_\gamma z_\gamma(1-y^*_\gamma) + (1-\frac{1}{u})\sum_\gamma a_\gamma(1-y^*_\gamma) - (1-\frac{1}{u})\sum_\gamma a_\gamma(1-y^*_\gamma)z_\gamma + \frac{1}{u}\sum_\gamma a_\gamma y^*_\gamma z_\gamma = (-\tilde{\Phi} + (1-\frac{2}{u})\sum_\gamma a_\gamma z_\gamma)\sum_\gamma(1-y^*_\gamma) + \tilde{\Phi} - \tilde{\Phi}\sum_\gamma y^*_\gamma z_\gamma - (2-\frac{3}{u})\sum_\gamma a_\gamma z_\gamma + (2-\frac{3}{u})\sum_\gamma a_\gamma y^*_\gamma z_\gamma + (1-\frac{1}{u})\sum_\gamma a_\gamma(1-y^*_\gamma) + \frac{1}{u}\sum_\gamma a_\gamma y^*_\gamma z_\gamma$.

The separation objective function (subject to the unit vector z) becomes $max_z(-\tilde{\Phi}+(1-\frac{2}{u})\sum_\gamma a_\gamma z_\gamma)\sum_\gamma(1-y^*_\gamma)+\tilde{\Phi}-\tilde{\Phi}\sum_\gamma y^*_\gamma z_\gamma-(3+u-\frac{3}{u})\sum_\gamma a_\gamma z_\gamma+2(1-\frac{1}{u})\sum_\gamma a_\gamma y^*_\gamma z_\gamma+(1-\frac{1}{u})\sum_\gamma a_\gamma(1-y^*_\gamma)+\sum_\gamma a_\gamma \tilde{q}^*_\gamma z_\gamma$. The z-coefficient with the maximal value gives j in the partition with the condition that an optimal objective value is positive. So, we are looking for index j^*, if any, which is related to the maximal value ($j^* = argmax_j$) among $a_j(1-\frac{2}{u})\sum_\gamma(1-y^*_\gamma)-\tilde{\Phi}y^*_j-(3+u-\frac{3}{u})a_j+2(1-\frac{1}{u})a_j y^*_j+a_j\tilde{q}^*_j$ with the condition that this value $> \tilde{\Phi}\sum_\gamma(1-y^*_\gamma) - \tilde{\Phi} - (1-\frac{1}{u})\sum_\gamma a_\gamma(1-y^*_\gamma)$.

The incomplete linking inequalities can be written in the form:

$$((1-\frac{1}{u})a_j - \tilde{\Phi})y_j + (1-\frac{1}{u})\sum_{l\neq j} a_l(1-y_l) + a_j\tilde{q}_j \leq (u+1)a_j$$

After introducing the characteristic vector z, the y_j-term becomes $\sum_\gamma z_\gamma((1-\frac{1}{u})a_\gamma - \tilde{\Phi})y^*_\gamma$; the $\tilde{q}$-term is $\sum_\gamma a_\gamma\tilde{q}^*_\gamma z_\gamma$; the right hand side is $(u+1)\sum_\gamma a_\gamma z_\gamma$; the y_l-term becomes $(1-\frac{1}{u})\sum_\gamma a_\gamma(1-y^*_\gamma)(1-z_\gamma)$.

Combining all terms together for the separation objective function (subject to the unit vector z), we have $\sum_\gamma z_\gamma((1-\frac{1}{u})a_\gamma - \tilde{\Phi})y^*_\gamma + (1-\frac{1}{u})\sum_\gamma a_\gamma(1-y^*_\gamma)(1-z_\gamma) + \sum_\gamma a_\gamma\tilde{q}^*_\gamma z_\gamma - (u+1)\sum_\gamma a_\gamma z_\gamma = \sum_\gamma z_\gamma((1-\frac{1}{u})a_\gamma - \tilde{\Phi})y^*_\gamma + (1-\frac{1}{u})\sum_\gamma a_\gamma - (1-\frac{1}{u})\sum_\gamma a_\gamma z_\gamma - (1-\frac{1}{u})\sum_\gamma a_\gamma y^*_\gamma + (1-\frac{1}{u})\sum_\gamma a_\gamma y^*_\gamma z_\gamma + \sum_\gamma a_\gamma\tilde{q}^*_\gamma z_\gamma - (u+1)\sum_\gamma a_\gamma z_\gamma = \sum_\gamma z_\gamma(2(1-\frac{1}{u})a_\gamma - \tilde{\Phi})y^*_\gamma + (1-\frac{1}{u})\sum_\gamma a_\gamma - (u+2-\frac{1}{u})\sum_\gamma a_\gamma z_\gamma - (1-\frac{1}{u})\sum_\gamma a_\gamma y^*_\gamma + \sum_\gamma a_\gamma\tilde{q}^*_\gamma z_\gamma$. The z-coefficient with the maximal value gives j in the partition with the condition that an optimal objective value is positive. So, we are looking for index j^*, if any, which is related to the maximal value ($j^* = argmax_j$) among $(2(1-\frac{1}{u})a_j - \tilde{\Phi})y^*_j - (u+2-\frac{1}{u})a_j + a_j\tilde{q}^*_j$ with the condition that this value $> -(1-\frac{1}{u})\sum_\gamma a_\gamma + (1-\frac{1}{u})\sum_\gamma a_\gamma y^*_\gamma$.

It is known that finding a maximal or minimal element in an array of J variables requires J-1 comparisons, and for finding two largest or smallest entries it is sufficient to take $J + \lceil log_2 J \rceil - 2$ comparisons (see, for example, Chapter 9 in Cormen et al. [10]).

We do not consider in details the modified incomplete linking inequalities because they deviate from our unifying separation framework due to the presence of the base elements.

3.5 Computational Experiments.

We perform computational experiments with paying special attention to the strong cutting planes of this chapter. Our computational platform is developed in C++. We use IBM/ILOG CPLEX 12.2 with Concert Technology for comparison. The results herein are performed on a desktop with 1.6 GHz Pentium 4 CPU, 1 Gb RAM, running Microsoft Windows XP.

Before providing the main results, we have a few remarks. First, as we mentioned above, mixed clique inequalities in this work $\tilde{q}_{ij} + \frac{u-1}{u} y_{ij} \geq \; u + 1$ strengthen inequalities (20), i.e. $\tilde{q}_{ij} + y_{ij} \geq u + 1$ in the y-coefficient. It is clear that we just have a priori reformulation for free. However, the importance of these inequalities is observable for relatively small values of parameter u only. So, for the sake of purity of the experiments, we work with our initial non-reformulated problem, but avoid small u. Its minimal for the feasibility of the problem value has to be calculated precisely, as explained in Section 3.2. Also, with help of this parameter, we embed our pre-processing technique, i.e. y_{ij} is fixed at 1 if $\tilde{W}_{ij} \geq a_{ij}$. For brevity, we took positive a_{ij} only. In addition, like in Chapter II, the choice of the objective function may influence the computational performance.

Example 3.3.1 (continued). For the "generic" objective function (just min $\sum_j t_j + \sum_i \sum_j y_{ij}$), the optimal value of the LP relaxation is 1.486 with t_1=0, t_2=$\frac{26}{35} \approx 0.743$,

t_3=0, y_{11}=0, y_{12}=$\frac{26}{35}$, y_{13}=0, $\tilde{q}_{11}$=5, $\tilde{q}_{12}$=$\frac{149}{35} \approx 4.257$, $\tilde{q}_{13}$=5). This point can be cut off by either strong inequality listed above: the second mixed clique, first or third star-clique, and the incomplete linking inequalities. The optimal objective value of this problem is 2 with multiple optimal solutions t_1=0, t_2=1, t_3=0, y_{11}=0, y_{12}=1, y_{13}=0, $\tilde{q}_{11}$=$\tilde{q}_{13}$=5, $\tilde{q}_{12}$ is in the range between $\frac{17}{4}$=4.25 and $\frac{149}{35} \approx 4.257$. If we add a q-term in the objective function $-\frac{1}{(u+1)IJ}\sum_i \sum_j \tilde{q}_{ij}$, the qualitative picture of solutions remains the same with one exception: now the optimal value is 1.05 with only one value of $\tilde{q}_{12}$=4.25. In the main experiments, we continue to work with this 3-term objective function.

We create a series of challenging instances with deterministic matrices using the following formula: $a_{ij} = a_{base} - 0.01 * step * i + step * j$ (in the loop $i = 0...I-1$, $j = 0...J-1$) with $a_{base} = 0.5, step = 0.05$.

We run CPLEX in a usual manner with its preprocessor and compare its performance with the situation when CPLEX is enhanced with both our preprocessor and strong cuts. The computational results are summarized in Table 19. They demonstrate the importance of strong cuts in these structured problems. The first two columns provide the problem size and characteristics. This is followed by the total time elapsed to solve the instances without, or with the help of our preprocessor and added cuts. The column "Speedup" calculates the improvement in CPU time of CPLEX with the usercutcallbacks. We observe good speedup gain as the size of instances increases. The last column provides the number of applied user cuts.

In conclusion, the polyhedral results obtained in this chapter can be tried to extend for more specific cases. The mixed hyperedge method can be studied more for applying to other structured problems. Some "degrees of freedom" are remained when we use the convexification outcome to solve MINLPs, and computational experiments may suggest some feedback how to utilize the q-terms in the objective function.

Table 19: Results of Experiments

I_T x J	u	Total Time Elapsed CPU_1	with user cuts CPU_2	Speedup CPU1/CPU2	# of user cuts applied
15 x 8	30	3.41 sec	2.62 sec	1.30	49
16 x 8	30	14.56 sec	6.59 sec	2.21	53
17 x 7	30	24.38 sec	9.84 sec	2.48	61
18 x 8	30	580.16 sec	130.89 sec	4.43	70
19 x 9	30	807.94 sec	177.84 sec	4.54	92
20 x 9	30	14660.9 sec	1398.23 sec	10.49	110

CHAPTER IV

APPLICATION OF THEORY: MANAGING GUEST FLOWS IN GEORGIA AQUARIUM

4.1 Introduction

In April 2011 the Georgia Aquarium, located in the downtown area of Atlanta, opened a new exhibit-show named Dolphin Tales after three years of building a theater and atrium area for that show. Before the opening, the Georgia Tech team was invited to study and predict how the new exhibit would impact the guest flows and how to optimize the operations logistics, efficiency and show schedules. Almost all recommendations, summarized in the Interfaces' publication (Lee et al. 2012 [25]), were implemented for the initial period of operation. To predict the guest flow pattern, the team had to study the entire aquarium, collect data, and develop their special software. Any amusement and attraction businesses are always in need of innovations in the form of new exhibits, updates, changes. The Aquarium had new events after that opening, and this study is dedicated to the guest flows to the scheduled shows in new circumstances.

The aquarium is claimed as the largest in the world with more than 8.5 million gallons of water and 120,000 animals. It accepts thousands of visitors every day with multiple numbers on weekends. The fundamental difficulty of such studies comes from incorporating new units into the existing facility with extremely hard layout restrictions. Figure 12 contains freely distributed for visitors and available online map-scheme of the Georgia Aquarium. It combines numerous points of interest, which are spread across two floors.

Five stationary exhibits, or theme-based wings (Georgia Explorer, River Scout,

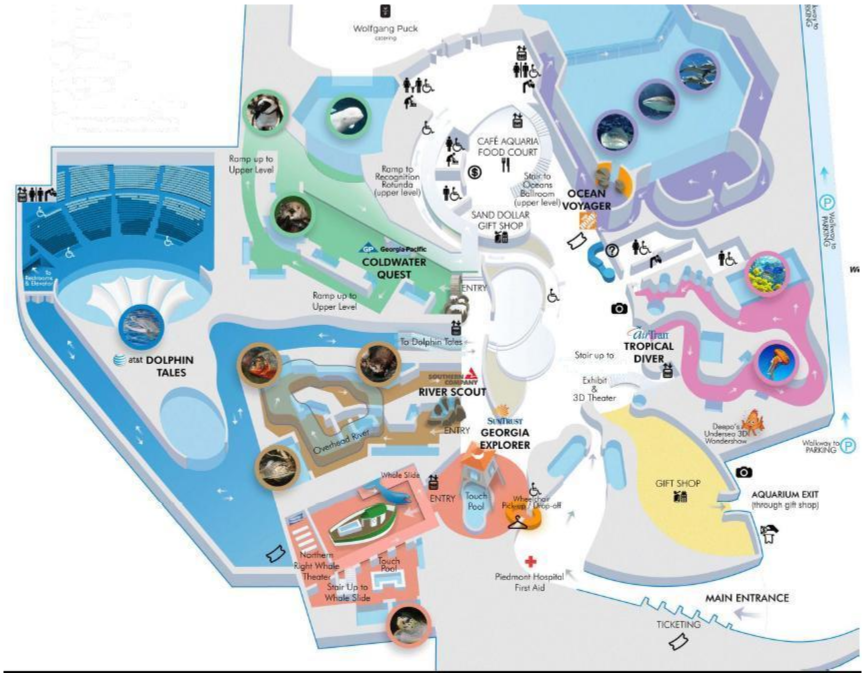

Figure 12: Georgia Aquarium

Figure 13: Central area with stairs to the 3D Theater

Figure 14: Queue tail to Dolphin Tales

Coldwater Quest, Ocean Voyager, and Tropical Diver) have entrances on the first floor via the main lobby (central area). They are supplemented with the gift shop, cafeteria, and other facilities. We pay special attention to two sites on the second floor: 3D Theater (precisely above the gift shop) and the Dolphin Theater on the left. Those two are special because they are related to the scheduled shows and located in isolation. It means that they are accessible for casual visitors from the central area of the first floor with returning back in the same way. For example, to attend the Deepo's Undersea 3D Wondershow (or "3D Deepo Show", for short) in the 3D Theater, spectators use the stairs (see Figure 13). In order to get the Dolphin Theater after that, they return to the central area via the same stairs and need to use two escalators (Figure 14) to the second floor.

It is necessary to emphasize that all business innovations may happen only inside of the very restricted space in our case of study. Scheduled events are especially sensitive to such modifications. Comparing to the time of the previous work, Georgia Aquarium management substituted one separately paid exhibit (Planet Shark) with another free exhibit, and did experiments with scheduling and ticketing for the shows. The visitors are no longer in need of purchasing extra tickets because there is a

Figure 15: General pass and ticket to Dolphin Tales

possibility to get everything with a general admission pass (Figure 15). Following the previous recommendation of at least a 90-minute interval between the Dolphin Tales (or "Dolphin Show"), the aquarium managers are still regulating the attendance and the arrivals to this attraction site. All customers receive an extra paper or pass with the precise time of the show (chosen in advance), plus the written advice to arrive about 30 minutes prior their show time and the warning about the door closing at the prompt time of the show.

About one hour before a scheduled Dolphin Show, both escalators become oriented upward, and visitors of the Dolphin Theater (with 1,865 seats) go through the sequence of three corridors. After about two thirds of their way, people meet extra check-in service and more restrictions. There are no possibilities to open the doors early, no assigned seats in the theater, and not enough waiting space on the second floor. Besides, the check-in service before the entrance on the 2nd floor cannot be started even 45 minutes before the show time. Figure 16 and 17 illustrate the area near the Dolphin Theater at the moment of check-in preparation and during the

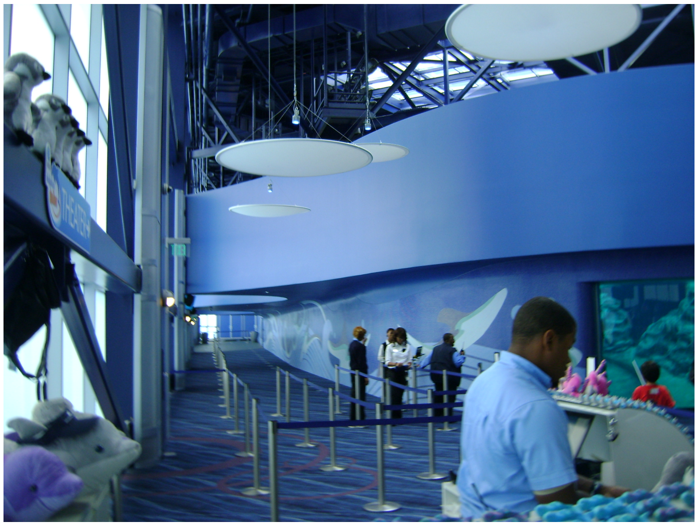

Figure 16: Closed Dolphin Theater

active check-in, respectively.

Figure 17: Dolphin Theater is open

Those visitors leave the Dolphin Theater through the same set of doors and corridors, and the allowed capacity of the area on the second floor is only about 30% of the theater capacity. The situation suggests using an evacuation-type model because the people have to exit the site quickly and have a very short period of time to leave the limited area on the second floor. In addition to the same way as for the arrival, a special exit on the second floor is opened for a part of the departure stream, but it is not for regular visits and can be used for a few minutes only in one direction. Figure 18 partially illustrates what a spectator can see after traversing the first corridor from the theater: the notice about the maximum occupancy in the area and the doors to the special exit. But most of visitors have to turn left to corridor 2. Figure 19 shows the scheme of the corridors.

Figure 18: Restricted area and extra exit

The special exit leads directly to the 3D Deepo Show place. This show is scheduled in even intervals and provides a batch service according to its limited capacity or just serves all arrived at a particular time people, if their number is less than the capacity. There are no assigned seats in the 3D Theater, but different doors for entrance and exit. Tickets are not issued, but it is possible to count visitors because special glasses are distributed before entering the waiting area. People drop those glasses to boxes after the show.

Figure 20 provides a snapshot example of the scheduled shows. Because of at least a 90-minute interval between the Dolphin Shows and no interaction between their spectators are expected, it is sufficient to take only one of them in our study, for example, scheduled at the noon. However, it is possible to extend our model for the whole day if the final expected number of visitors for each show is known in

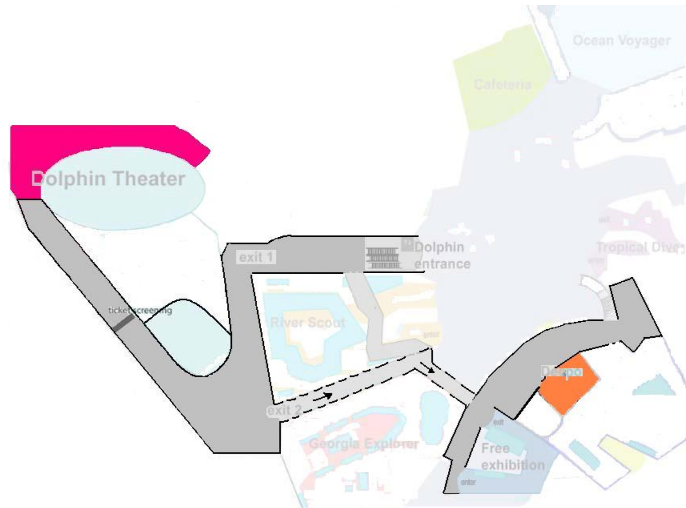

Figure 19: Corridors for visitors

advance. A natural "remedy" for the area limitation on the second floor is temporary blocking the escalators. However, waiting outside may impact the work of the entire aquarium and congest the first floor. In the worst case, the entire aquarium can be in the deadlock because uncontrolled people may create independently one line to the left escalator with blocking the left hallway and another line to the right escalator (as on Figure 14) with the tail of the queue towards the stairs, blocking the right hallway and visitors of the 3D Show. As Figure 20 suggests, there are two possible time periods in risk: one between 11:15 and 11:30 (with the growing arrival process to the Dolphin Show, incoming people to the 11:30 3D show, and outgoing people from the previous 15-minute 3D show), and the other starting at about 12:45 (when the people going from the Dolphin Theater may collide the visitors of the 3D Theater in both directions and on both floors).

We propose stochastic models for all three major processes: arrivals to the Dolphin Theater, departures from it, and arrivals to the 3D Theater. In general, when the demand for shows is relatively high, all three become interconnected and require a unified approach. Our goals are to prevent, control, and mitigate congestions, flow collisions and undesired shifts in demand for shows (upon possibilities, taking

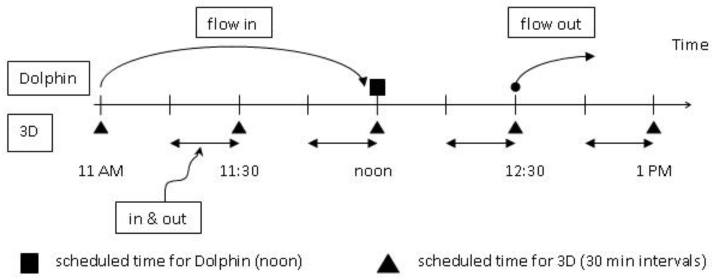

Figure 20: Scheduled shows: Dolphin Tales vs. Deepo 3D

into account existing space restrictions) with minimizing operating costs. Thus, we develop a large-scale Mixed Integer Nonlinear Programming (MINLP) problem of the whole picture related to the scheduled shows and solve it.

The outline of the rest of this chapter is as follows. In Section 4.2, we present a rush-hour-type model for the Dolphin Show arrival with estimation the length of the queue in the restricted area on the second floor and the tail of the queue on the first floor. Section 4.3 introduces an evacuation-type model for the Dolphin Show departures. For the 3D Deepo Show, we consider an equilibrium arrival model where customers minimize their waiting time in Section 4.4. Section 4.5 compiles the constraints of all three stochastic submodels together with their interactions into formulation of the large scale Mathematical Programming problem, containing linear and signomial term. Finally, our computational experience and managerial insights are presented in Section 4.6.

4.2 Arrival to the Dolphin Show Submodel

This section and the next one describe modeling arrivals to the Dolphin Show and departures after it, respectively. Visitors use the same corridors, but move in the opposite directions. With known arrival/exit rates and finite population $\tilde{N}$ (i.e. the number of the sold tickets for the particular show scheduled at time t^*), we build approximation models for nonstationary queueing systems. In both cases, a time interval is divided into periods, which are analyzed with stationary approximations. Estimated queue lengths are the major characteristics of the models and parts of the capacity constraints. Recall the maximum l_{max} people are allowed in the corridors near the Dolphin Theater.

We consider the arrival process to the Dolphin Show as $M(t)/G/c(t)$ queueing system during one hour before t^*. Customers are served according to the first-come first-served discipline. Time-dependent number $c(t)$ defines check-in counters-servers. At least one of them becomes open at some time t_{open} around 30 minutes before t^*, and check-in service remains available with at least one open server until t^*. The system assumes a nonhomogeneous Poisson arrival process with instantaneous arrival rates λ_i, $i = 1...T$. In other words, the time interval of interest (60 minutes before t^*) is divided into periods $i = 1...T$ with constant arrival rates and numbers of opened servers within each period i. Piecewise constant arrival rates are measured and given exogenously with observation that the arrival rate function $\lambda(t)$ grows approximately 30 minutes and decreases after that with boundary conditions $\lambda(0) = \lambda(T) = 0$. The check-in service time is considered as generally distributed random variable with rate μ and the squared coefficient of variation (i.e. the ratio of the variance to the squared mean) CV^2 identical for all opened servers during the whole arrival process. Finite population of customers $\tilde{N}$ is incorporated in the arrival rate function. For example, it is possible to consider the quadratic piecewise approximation $\lambda_i = \tilde{N}(-\frac{6}{T^3}i^2 + \frac{6}{T^2}i)$. So, we consider the queueing system with "infinite" population and waiting room without loss of generality.

We utilize a stationary backlog-carryover (SBC) approach (Stolletz (2008) [38], Stolletz (2011) [39]) with the goal to receive and use the expected queue length in each period i. This approach has three stages.

First, it applies a stationary $M/G/c_i/c_i$ loss model for period i with constant number of servers $0 \leq c_i \leq c_{max}, i = 1...T$. Artificial arrival rate $\tilde{\lambda}_i$ is defined as $\tilde{\lambda}_i = \lambda_i + b_{i-1}$, $i = 1...T$, where b_i is the backlog generated from artificially blocked customers, usually $b_0 = 0$.

Let $P_i(B)$ be the blocking probability. Then

$$b_i = \tilde{\lambda}_i P_i(B) = \tilde{\lambda}_i \frac{(\tilde{\lambda}_i/\mu)^{c_i}}{c_i! \sum_{k=0}^{c_i} \frac{(\tilde{\lambda}_i/\mu)^k}{k!}}$$

This number of artificially blocked customers is carried over as additional arrivals in the subsequent period $i+1$ and allows to calculate expected utilizations for each period as $E[U_i] = \frac{\tilde{\lambda}_i(1-P_i(B))}{c_i\mu} = \frac{\tilde{\lambda}_i - b_i}{c_i\mu}$, $c_i \geq 1$.

Second, it brings the expected utilization $E[U_i]$ to a stationary $M/G/c_i$ waiting queueing system. The modified arrival rate λ_i^{MAR} serves as an input to the stationary queueing model and defined in such a way that the approximated utilization $E[U_i]$ is achieved. In our case, $\lambda_i^{MAR} = c_i\mu E[U_i] = \tilde{\lambda}_i - b_i$.

Third, $M/G/c_i$ queueing system with the modified arrival rate λ_i^{MAR} is considered with a stationary approximation for the expected waiting time $E[W]$ and the expected queue length $E[Q]$.

For example, we can use the well-known Cosmetatos' approximation (Cosmetatos (1976) [11]):

$E[W_{M/G/c_i}] \approx CV^2 E[W_{M/M/c_i}] + (1 - CV^2) E[W_{M/D/c_i}]$,

and $E[Q_{M/G/c_i}] = \lambda_i^{MAR} E[W_{M/G/c_i}]$

The values of $E[W_{M/M/c_i}]$ and $E[W_{M/D/c_i}]$ are exact from the theory ($c \geq 1$, steady-state cases):

1) for $M/M/c$ with parameters $\hat{\lambda}$ and $\hat{\mu}$:

$$E[W_{M/M/c}] = P_o \frac{(\hat{\lambda}/\hat{\mu})^c \hat{\mu}}{(c-1)!(c\hat{\mu} - \hat{\lambda})^2},$$

where

$$P_o = (\sum_{k=0}^{c-1} \frac{(\hat{\lambda}/\hat{\mu})^k}{k!} + \frac{(\hat{\lambda}/\hat{\mu})^c \hat{\mu}}{(c-1)!(c\hat{\mu} - \hat{\lambda})})^{-1}.$$

($\hat{\lambda} = \lambda^{MAR}$ in the respective period in our case)

2) for $M/D/c$ with parameters $\hat{\lambda}$ and $\hat{\mu}$

$$E[W_{M/D/c}] = \frac{1}{2}[1 + (1 - \frac{\hat{\lambda}/\hat{\mu}}{c})(c-1)\frac{\sqrt{4+5c} - 2}{16\hat{\lambda}/\hat{\mu}}]E[W_{M/M/c}]$$

So, $E[W_{M/G/c}] \approx E[W_{M/M/c}](\frac{1+CV^2}{2} + \frac{(1-CV^2)(1-\frac{\hat{\lambda}/\hat{\mu}}{c})(c-1)(\sqrt{4+5c}-2)}{32\hat{\lambda}/\hat{\mu}})$.

These formulas for $E[W]$ (and automatically for $E[Q]$) require the stability condition $\hat{\lambda} < c\hat{\mu}$, which is not guaranteed in our situation. Moreover, servers do not work in first minutes. We assume getting stabilization only closer to the end of the hour in consideration. So, we also need a "fluid" alternative approximation of the expected queue length $E[Q_{M/M/c}]$ at the end of period i as the backlog rate b_i multiplied by the period length $\Delta_i = t_i - t_{i-1}$ (Stolletz (2008) [38]). We consider $\Delta = \Delta_i$ $\forall i$. If we assume that the arrival and service characteristics are already adjusted to the period length (rather than being per minute), it means that we have $\Delta = 1$ and can disregard it. Thus, $b_i \leq l_{max}, i = 1...T-1$ (as $E[Q] \leq l_{max}$ always), and $E[Q_{M/G/c_T}] \approx b_T(\frac{1+CV^2}{2} + \frac{(1-CV^2)(1-\frac{\lambda_T^{MAR}/\mu}{c_T})(c_T-1)(\sqrt{4+5c_T}-2)}{32\lambda_T^{MAR}/\mu}) \leq l_{last}$ are added to the constraints, where l_{last} is an input parameter for the desired number of remaining customers in the last time period. For example, if $T = 60$ and $l_{last} = 1$, it means the we expect one visitor in the queue (in the steady state sense) at the last minute before the start of the show.

4.3 Departure from the Dolphin Show Submodel

This section models the departure process from the Dolphin Theater starting at time $t_{dep} = t^* + t_{tales}$, where t_{tales} is duration of the show. In contrast to the process in the previous section, the same $\tilde{N}$ visitors have the time restriction to empty the theater during T_d minutes (much less than 60) in addition to the space limitation l_{max} in the same corridors. The previous study dictates using an additional special exit towards the first floor for a limited period of time T_{ex} after t_{dep}.

The nature of the process suggests coupling stationary and fluid based approximations. SBC was used in the previous section. More basic approaches include simple stationary (SSA) and pointwise stationary (PSA) approximations, see Green and Kolesar (1991) [19] and Green et al. (1991) [20]. SSA ignores nonstationarity by considering stationary M/M/c queueing models with the arrival rate averaged over the whole time interval of interest. PSA uses stationary queueing models with instantaneous rates at each time point of the period in consideration.

An evacuation-type model by Smith (see Bedell and Smith (2012) [3]), and model variants in references therein) is suitable as a stationary component in our approach. Smith's models deal with steady state situations. They have the following major assumptions: visitors departure according to a homogeneous Poisson process with rate λ_d, the traffic density determines the average walking speed leading to a state dependent service rate (decaying with increased traffic), the amount of available space is finite, and people can be considered as approximately uniformly distributed in rectangular corridors.

We consider a variant of linear decay of the walking speed. When n customers occupy a corridor, they have the average walking speed $V_n = \frac{V_1}{C}(C+1-n)$, $n = 1...C$, $V_{C+1} = 0$, where V_1 is a speed of a lonely customer. The corridor capacity is defined as $C = \gamma LW$, where γ is the density, i.e. the number of people per a squared meter leading to the congestion. L and W are the length and width of the corridor respectively. Service rate of each of n individuals in the corridor is $\hat{r}_n = \frac{V_n}{L}$ (i.e. average of the inverse of the time it takes these individuals to traverse the length of the corridor). The overall service rate of this Erlang M/M/C/C system is $\mu_n = n\hat{r}_n$, and the state probabilities are $p_n = \frac{\lambda_d^n}{\mu_1\mu_2...\mu_n}p_0$, where $\frac{1}{p_0} = 1 + \sum_{n=1}^{C} \frac{\lambda_d^n}{\mu_1\mu_2...\mu_n}$, $n = 1...C$.

The probability of blocking p_C plays the most important role in future consideration. The mean number of customers in the system is $\sum_{n=1}^{C} np_n$.

It is shown (Cheah and Smith (1994) [8]) that this M/M/C/C model is stochastically equivalent to M/G/C/C one, which can be written as $p_n = \frac{(\lambda_d E(T_1))^n}{n!f(n)...f(2)f(1)}p_0$, with $\frac{1}{p_0} = 1 + \sum_{n=1}^{C} \frac{(\lambda_d E(T_1))^n}{n!f(n)...f(2)f(1)}$, $n = 1...C$, where $E(T_1) = \frac{L}{V_1}$ (the expected service time of a lone occupant in the corridor) and $f(n) = \frac{V_n}{V_1}$.

Fluid flow models are based on the flow conservation principle, i.e. the rate of change of a state variable (representing the average number of people) is equal to the difference between inflow and outflow: $\frac{dl(t)}{dt} = inflow(t) - outflow(t)$. Wang et al. (1996) [43] consider PSFFA (PSA + Fluid Flow Approximation) models for queueing systems with a single server. If the queue waiting space is infinite, then the $inflow(t)$ is the arrival rate at t and the $outflow(t)$ contains the product of the average service rate and the average utilization of the server $\rho(t)$. Determining an expression for $\rho(t)$ depends on the queue system and can be subtle. With the PSA approach, $\rho(t)$ is obtained at particular instants of time from $l(t)$ using the steady state ($\frac{dl(t)}{dt} = 0$) relationships and a repeated procedure over small time steps. In some cases, it is possible to find $\rho(t)$ as an explicit inverse function of $l(t)$. Otherwise (if a closed form of this inverse function is not available), the function can be determined numerically or by curve fitting. For queueing systems with finite capacity, $inflow(t)$ is the product of the arrival rate at t and the customer nonblocking probability $(1 - P_B(t))$; and $outflow(t)$ is the product of the average service rate and the probability of having a nonzero number of customers $(1 - P_0(t))$. After getting $\rho(t)$ as a function of $l(t)$, $P_B(t)$ and $P_0(t)$ are estimated as functions of $\rho(t)$.

Chen et al. (2013) [9] develop another fluid based approximation method (called B-PSFFA) with using PSA for a nonstationary multiserver Exponential-Erlang model $M(t)/E_k(t)/c(t)$ and adopting a stationary queueing model in the form of the Cosmetatos' approximation. They consider a flow balance $l_{j+1} = l_j + \lambda_j - v_j$ for the change of the queue length from time point j to time point $j+1$, the inequality for the departure $v_j \leq c(t)\rho_j\mu_j$, and ρ_j as an inverse function of l_j (containing the inverse expression of the Cosmetatos' approximation).

Alnowibet and Perros (2009) [1] consider loss queues with nonhomogeneous Poisson arrivals and exponential service times. They propose the fixed point approximation (FPA) method based upon the differential equation $\frac{dl(t)}{dt} = \tilde{\lambda}(t)(1-P_B(t)) - \tilde{\mu}l(t)$. The method iteratively approximate the blocking probability and $\rho(t) = \frac{l(t)}{1-P_B(t)}$, where $P_B(t)$ starts from 0 and then calculated as the Erlang loss equation (like that for the steady-state blocking probability) from the previous iteration. FPA was extended to general service time distributions by Izady and Worthinghton (2011) [22].

In our approach, we also combine steady state elements with flow approximations. We take discretization of the T_d-minute time interval and utilize the result from the preopening study that the departure process can be considered as having the homogeneous Poisson rate Λ until approximately the time moment $\frac{\tilde{N}}{\Lambda}$. We can assume that initially the blocking probability =0, and we can apply the steady state conditions after the moment of the expected traverse time of the first visitor. So, for the first corridor we consider $l_{j+1}^{(1)} = l_j^{(1)} + \Lambda(1 - BP_j^{(1)}) - v_j^{(1)}$ and apply the conservative bounds from the Smith's model: $BP_j^{(1)} \geq p_{C^{(1)}}$ and $v_j^{(1)} \leq \Delta_j l_j^{(1)} \frac{V_1}{C^{(1)}L^{(1)}}(C^{(1)} + 1 - l_j^{(1)})$. Because the parameters in Smith's models are in metric units and seconds, the time period Δ_j is measured in seconds as well. If we take V_1 per Δ_j rather than per second, it means taking $\Delta_j = 1$. The rate Λ is also adjusted per Δ_j. The special exit usage after corridor 1 modifies the expressions for corridor 2 with variable coefficients α_j (representing the part of visitors deviated via the special exit): $l_{j+1}^{(2)} = l_j^{(2)} + (1-\alpha_j)v_j^{(1)}(1 - BP_j^{(2)}) - v_j^{(2)}$ with $BP_j^{(2)} \geq p_{C^{(2)}}(\alpha_j)$. The steady state blocking probability from the Smith's model is a function of α_j. This function is found numerically and approximated with curve fitting.

4.4 Arrival to the 3D Deepo Show Submodel

This section introduces the arrival process to the 3D Deepo Show. It is based on the prototype model by Glazer and Hassin (1987) [17], also described in the book by Hassin and Haviv (2003) [21]. They consider a system like a public bus transportation (with finite capacity of N seats) in which the service starts at fixed evenly spaced times, regardless of the number of customers in the queue with first-come first-served discipline. So, if more than N people are present at the time of service, then N are served and the rest wait for future services. Arrivals in various periods (also called "cycles") between scheduled times are independent and identically distributed. It is assumed that the total number of arrivals within a cycle is a Poisson random variable. For the sake of simplicity, the time period (0,1] is chosen for consideration without loss of generality.

A customer decides when to arrive with the goal to minimize his or her expected waiting time. It is clear that in the case of the guarantee of having less than N customers it would be efficient to arrive closer to the end of the cycle (which is also "social equilibrium" desirable for the service management). However, the presence of a positive probability of having more than N waiting customers dictates the necessity of finding an individual equilibrium arrival time t_0 within the cycle. Glazer and Hassin construct numerically an equilibrium distribution $F(t)$ of customer arrivals, i.e. $F(t)$ is the probability that a random customer arrives at most t time units after the beginning of a cycle. By equilibrium definition for identical customers with the same information, there are no arrivals in $(0, t_0)$. Plus, the arrival process during $[t_0,\ 1]$ follows a nonhomogeneous Poisson process having the rate $\lambda(t) = \lambda_a \frac{dF(t)}{dt}$ with mean $\lambda_a < N$.

From equilibrium arguing, we have $t_0 = \sum_{j=0}^{N-1} r_j$, where r_j is the probability that j customers are in the queue just before a scheduled service. The values of r_j are obtained from the system of equations: $r_0 = q_0 \sum_{i=0}^{N} r_i$, $r_j = q_j \sum_{i=0}^{N} r_i +$

$\sum_{i=1}^{j} q_{j-i} r_{N+i}$, $j > 0$, where the values of q_j (probability that j new customers arrive in a cycle) are exogenously given.

This system of equations can be solved by successive approximations after taking a reasonable bound of the counter j and assigning values to the initial vector r^0, that sum to 1. The left hand side is just the vector r^{k+1} of the $(k+1)$th iteration with r^k in the right hand side. Glazer and Hassin (1987) [17] demonstrate that taking $j = 1...3N+1$ in their example for $N = 50$ gives sufficiently accurate results.

For the real capacity of N seats in the 3D Deepo Theater, we calculate numerically an equilibrium arrival time as a function of an expected arrival rate, taking q_j according to the Poisson distribution. Then curve fitting approximates the obtained function.

4.5 Large Scale Mathematical Programming Problem Formulation

This section demonstrates interconnection of all three stochastic models described in the previous sections. The large scale Mixed Integer Nonlinear Programming (MINLP) problem of queueing control is derived using the provided lists of input parameters and variables.

Input data and parameters:

t^*: starting time of the particular Dolphin Show

$\tilde{N}$: number of the sold tickets for the Dolphin Show scheduled at time t^*

$T \geq 12$: number of time intervals for discretization of the Dolphin Show arrival period of one hour before t^*

λ_i, $i = 1...T$: arrival rates to the Dolphin Show

$bound_\mu$: min average service time of a check-in server

c_{max}: max number of check-in servers to the Dolphin Show

b_0: initial backlog in the arrival to the Dolphin Show

l_{max}: max occupancy in the area near the Dolphin Theater

l_{last}: expected number of remaining customers in the last time period of arriving to the Dolphin Show

l_{lobby}: max occupancy in the area before the escalators to recommend the control of the 1st floor

N: capacity of the 3D Deepo Theater

T_{3D}: number of 3D Deepo Shows in consideration

$\hat{\lambda}_t$: expected arrival rate to the 3D Deepo Show scheduled at time t

d_{del}: expected part of delayed demand to 3D Deepo Shows

T_{dep}: number of time intervals for discretization of the Dolphin Show departure period started at t^* Λ: stationary exit rate from the Dolphin Theater

V_1: speed of a lonely customer during departure from the Dolphin Theater

$C^{(1)}, C^{(2)}$: capacity of corridors 1 and 2 respectively

$L^{(1)}, L^{(2)}$: length of corridors 1 and 2 respectively

τ_{cr}: threshold level of the equilibrium arrival time to the 3D Deepo Show for the special exit restrictions

$k_1, k_2, k_3, k_4, k_5, k_6$: unit costs for check-in servers, escalator blocking, the main lobby, the 3D Theater, the Dolphin Theater departure, and the special exit control, respectively

Variables:

b_i, $i = 1...T$: backlog from artificially blocked customers in the Dolphin Show arrival

$\tilde{\lambda}_i$, $i = 1...T$: artificial arrival rates to the Dolphin Show

μ: service rate of one check-in server to the Dolphin Show (adjusted to the time units)

σ: deviation in service time of one check-in server in the Dolphin Show arrival (adjusted to the time units)

CV^2: squared coefficient of variation of one check-in server to the Dolphin Show (adjusted to the time units)

c_i, $i = 1...T$: number of open check-in servers to the Dolphin Show

$y_{0i}, y_{1i}, y_{2i}, ...y_{c_{max}i}$, $i = 1...T$: binary variables related to the number of open servers to the Dolphin Show (=1, if the respective number of servers is used)

$\lambda_M = \tilde{\lambda}_T - b_T$: modified arrival rate to the Dolphin Show in the last time period

z_T: auxiliary variable, $z_T^2 = 4 + 5c_T$

γ_{esc_i}, $i = 1...T$: binary variables related to blocking the escalators (=1, if yes)

γ_{con_i}, $i = 1...T$: binary variables related to recommendation of the first floor control (=1, if yes)

γ_i, $i = 1...T$: binary variables (=1, if barriers are installed, i.e. the recommendation of control is implemented)

γ_{del_t}, $t = 1...T_{3D}$: binary variables related to performing the main lobby control (=0, if implemented, i.e. no delayed demand to 3D Deepo)

$\delta_{a25}, \delta_{a45}$: auxiliary binary variables (=1, if $\sum_{i=1}^{a_{del_{25}}} \gamma_{con_i} > 0$ and $\sum_{i=1}^{a_{del_{45}}} \gamma_{con_i} > 0$ respectively, where $a_{del_{25}}$ is the smallest integer such that $\frac{5}{12}T \leq a_{del_{25}} < \frac{1}{2}T$, and $a_{del_{45}}$ is the rounding of $\frac{3}{4}T$)

$\delta_{nobar25}, \delta_{nobar45}$: auxiliary binary variables (=1, if $\sum_{i=1}^{a_{del_{25}}} \gamma_i = 0$ and $\sum_{i=1}^{a_{del_{45}}} \gamma_i = 0$ respectively)

$\tilde{\tilde{\lambda}}_t$: modified expected arrival rate to the 3D Deepo Show scheduled at time t

τ_t: equilibrium arrival time to the 3D Deepo Show scheduled at time t

λ_{trun}: auxiliary arrival rate to the 3D Deepo Show, i.e. its truncation to [220;250]

y_{1tr}, y_{2tr}: binary variables to provide $\lambda_{trun} \in [220; 250]$

$\delta_{\lambda 220}, \delta_{\lambda 250}$: auxiliary binary variables (=1, if $\tilde{\tilde{\lambda}}_{t^*+2} > 220$ and $\tilde{\tilde{\lambda}}_{t^*+2} \leq 250$ respectively)

$\delta_{\omega_t}, \delta_{\nu_{t+1}}$, $t = 1...T_{3D} - 1$: auxiliary binary variables (=1, if $\tilde{\tilde{\lambda}}_t > 250$ and $\tilde{\tilde{\lambda}}_{t+1} > 250$ respectively)

$\gamma_{3D_{t+1}}$: binary variables (=1 for the control of the area near 3D Theater)

λ_{ex_j}, $j = 1...T_{dep}$: departure rate from the Dolphin Theater

$l_j^{(1)}$, $l_j^{(2)}$, $j = 1...T_{dep}$: expected number of people exiting the Dolphin show at time j in corridors 1 and 2 respectively

$BP_j^{(1)}$, $BP_j^{(2)}$, $j = 1...T_{dep}$: blocking probabilities in corridors 1 and 2 respectively

$v_j^{(1)}$, $v_j^{(2)}$, $j = 1...T_{dep}$: number of exiting people from corridors 1 and 2 respectively

γ_{dep_j}, $j = 1...T_{dep}$: binary variables (=1, if corridor 1 is controlled during departure from the Dolphin Show)

α_j, $j = 1...T_{dep}$: part of people using the special exit ($0 \leq \alpha_j \leq 1$)

$\tilde{\alpha}_j$, $j = 1...T_{dep}$: auxiliary variables, related to α_j

$\hat{\alpha}_j$, $j = 1...T_{dep}$: binary variables (=1, if the special exit is open)

$\delta_\alpha, \delta_{\alpha\alpha}$: auxiliary binary variables (=1, if and only if $\sum_{j=\lfloor \frac{2}{3} T_{dep} \rfloor + 1}^{T_{dep}} \alpha_j = 0$ and $\sum_{j=\lfloor \frac{1}{2} T_{dep} \rfloor + 1}^{T_{dep}} \alpha_j = 0$ respectively)

δ_τ: auxiliary binary variable (=1, if $\tau_{t^*+2} \leq \tau_{cr}$).

The problem has the following constraints.

Recall the backlog formula (with $b_0 = 0$ as a natural input)

$\tilde{\lambda}_i = \lambda_i + b_{i-1}, i = 1...T.$

The check-in service rate μ and CV^2 are considered as variables having bounds for the performance (rather than parameters). In this work, the service time is assumed having uniform distribution $U[\frac{1}{\mu} - \sigma, \frac{1}{\mu} + \sigma]$ with variables μ and σ satisfying the constraints $\frac{1}{\mu} \geq bound_\mu$ and $\frac{1}{\mu} - \sigma \geq \frac{1}{3} bound_\mu$. Thus, $CV^2 = \frac{1}{3}\sigma^2\mu^2$.

Having a small upper bound of the number of check-in servers ($c_{max} = 3$) and introducing binary variables y_{0i}, y_{1i}, y_{2i}, y_{3i} for the number of open servers in each period of time, we can transform the expression $b_i = \tilde{\lambda}_i \frac{(\tilde{\lambda}_i/\mu)^{c_i}}{c_i! \sum_{k=0}^{c_i} \frac{(\tilde{\lambda}_i/\mu)^k}{k!}}$ into the constraints in the signomial form:

$$b_i y_{0i} + b_i\tilde{\lambda}_i y_{1i} + b_i \mu y_{1i} + 2b_i\mu^2 y_{2i} + 2b_i\tilde{\lambda}_i\mu y_{2i} + b_i\tilde{\lambda}_i^2 y_{2i} + 6b_i\mu^3 y_{3i} + 6b_i\tilde{\lambda}_i\mu^2 y_{3i} + 3b_i\tilde{\lambda}_i^2\mu y_{3i} + b_i\tilde{\lambda}_i^3 y_{3i} - \tilde{\lambda}_i y_{0i} - \tilde{\lambda}_i^2 y_{1i} - \tilde{\lambda}_i^3 y_{2i} - \tilde{\lambda}_i^4 y_{3i} = 0, i = 1...T$$

together with

$$y_{0i} + y_{1i} + y_{2i} + y_{3i} = 1,\ i = 1...T$$

$c_i = y_{1i} + 2y_{2i} + 3y_{3i},\ i = 1...T$

We also require that the Dolphin Theater is closed at least first 20 minutes (i.e. about $\lceil T/3 \rceil$) after starting the arrival of people and at least one server is open during 30 minutes before the Dolphin show.

$y_{0i} = 1,\ i = 1...\lceil T/3 \rceil$

$y_{0i} = 0,\ i = \lfloor T/2 \rfloor + 1...T$

After denoting $\lambda_M = \lambda_T^{MAR} = \tilde{\lambda}_T - b_T$, $z_T = \sqrt{4+5c_T}$, and plugging $CV^2 = \frac{1}{3}\sigma^2\mu^2$, the constraint $b_T(\frac{1+CV^2}{2} + \frac{(1-CV^2)(1-\frac{\lambda_T^{MAR}/\mu}{c_T})(c_T-1)(\sqrt{4+5c_T}-2)}{32\lambda_T^{MAR}/\mu}) \le l_{last}$ is transformable into the signomial form:

$\frac{9}{16}b_T + \frac{7}{48}b_T\sigma^2\mu^2 + \frac{1}{32}\lambda_M^{-1}b_T\mu c_T z_T - \frac{1}{32}b_T z_T - \frac{1}{96}\lambda_M^{-1}b_T\sigma^2\mu^3 c_T z_T + \frac{1}{96}b_T\sigma^2\mu^2 z_T - \frac{1}{16}\lambda_M^{-1}b_T\mu c_T + \frac{1}{48}\lambda_M^{-1}b_T\sigma^2\mu^3 c_T - \frac{1}{32}\lambda_M^{-1}b_T\mu z_T + \frac{1}{32}c_T^{-1}b_T z_T + \frac{1}{96}\lambda_M^{-1}b_T\sigma^2\mu^3 z_T - \frac{1}{96}c_T^{-1}b_T\sigma^2\mu^2 z_T + \frac{1}{16}\lambda_M^{-1}b_T\mu - \frac{1}{16}c_T^{-1}b_T - \frac{1}{48}\lambda_M^{-1}b_T\sigma^2\mu^3 + \frac{1}{48}c_T^{-1}b_T\sigma^2\mu^2 \le l_{last}$

together with

$z_T^2 = 4 + 5c_T$

$\lambda_M = \tilde{\lambda}_T - b_T$

The length of the queue to the check-in of the Dolphin Show may exceed l_{max}. In this case, the tail of this queue is not allowed to be on the second floor, and the escalators are subject to blocking ($\gamma_{esc_i} = 1$). Furthermore, the space of the main lobby on the first floor is also limited (by l_{lobby}), and the visitors may block the hallways and the stairs towards the 3D Deepo Theater. In that case, it is recommended to spend resources for controlling the first floor ($\gamma_{con_i} = 1$), for example, in the form of setting barriers ($\gamma_i = 1$). So,

$b_i - l_{max} \le \tilde{N}\gamma_{esc_i},\ i = 1...T$

$b_i - l_{max} - l_{lobby} \le \tilde{N}\gamma_{con_i},\ i = 1...T$

$\gamma_{con_i} \le \gamma_{esc_i},\ i = 1...T$

$\gamma_i \le \gamma_{con_i},\ i = 1...T.$

To connect the arrival processes to the Dolphin Theater and the 3D Deepo Theater, we use the index $t = 1...T_{3D}$ in this section for the scheduled times of the 3D Deepo Show. In a casual week day, $t = 1$ is related to the 10:30 show, $t = 2$ to the 11 o'clock show, and so on until $t = 13$ to the 16:30 show. On weekends, $T_{3D} > 13$. Now t^* is related to both Dolphin and 3D Deepo Shows in consideration scheduled simultaneously. Respectively, $t^* - 1$ and $t^* + 1$ indicate the previous and the next 3D Deepo Shows. Because we consider only one particular Dolphin Show (with the final number of the sold tickets at one hour before t^*), without loss of generality, we use T_{3D} for indication a few 3D shows with inclusion the one scheduled at t^*. Demand for a particular 3D show may be delayed if the potential spectators see the blocked stairs by a long tail of the queue to the escalators. If the control mode was recommended, but not implemented (i.e. $\sum_i \gamma_{con_i} > 0$ and $\sum_i \gamma_i = 0$), the delay of demand for 3D is inevitable (and respective $\gamma_{del_t} = 1$). Approximately, the time spots of 35 and 15 minutes before t^* serve as indicators of possible delays. For $T = 60$, we accept that if $\sum_{i=1}^{25} \gamma_{con_i} > 0$ and $\sum_{i=1}^{25} \gamma_i = 0$ then $\gamma_{del_{t*}} = 1$. If $T \neq 60$, the number 25 is substituted by the smallest integer $\frac{5}{12}T \leq a_{del_{25}} < \frac{1}{2}T$.

Thus,

$\sum_{i=1}^{a_{del_{25}}} \gamma_{con_i} - a_{del_{25}} \delta_{a25} \leq 0$

$\sum_{i=1}^{a_{del_{25}}} \gamma_i + \delta_{nobar25} \geq 1$

$-\delta_{a25} + \gamma_{del_{t*}} \leq 0$

$-\delta_{nobar25} + \gamma_{del_{t*}} \leq 0$

$\delta_{a25} + \delta_{nobar25} - \gamma_{del_{t*}} \leq 1$

Similarly, if, in addition, $\sum_{i=1}^{45} \gamma_{con_i} > 0$ and $\sum_{i=1}^{45} \gamma_i = 0$, then $\gamma_{del_{t^*+1}} = 1$ as well, and the number 45 would be substituted by an integer number $a_{del_{45}}$ as rounding of $\frac{3}{4}T$. So,

$\sum_{i=1}^{a_{del_{45}}} \gamma_{con_i} - a_{del_{45}} \delta_{a45} \leq 0$

$\sum_{i=1}^{a_{del_{45}}} \gamma_i + \delta_{nobar45} \geq 1$

$-\delta_{a45} + \gamma_{del_{t^*+1}} \leq 0$

$-\delta_{nobar45} + \gamma_{del_{t^*+1}} \leq 0$

$\delta_{a45} + \delta_{nobar45} - \gamma_{del_{t^*+1}} \leq 1$

We assume that the expected arrival rate $\hat{\lambda}_t$ to the 3D Deepo Show scheduled at time t (given exogenously, e.g. proportionally to the arrival rates to the whole aquarium measured from hour to hour) is modified with counting postponed previous demand using parameter d_{del}. According to the observation, the visitors of 3D shows do not wait near the door more than one show. The local character of our analysis around time t^* sets variables γ_{del_t} to 0 beyond $t = t^*$ and $t = t^* + 1$. Therefore, (with $\tilde{\tilde{\lambda}}_0 = 0$):

$\tilde{\tilde{\lambda}}_t = \hat{\lambda}_t + d_{del}\tilde{\tilde{\lambda}}_{t-1}\gamma_{del_t}$, $t = 1...T_{3D}$

$\gamma_{del_t} = 0$, $t \neq t^*$, $t \neq t^* + 1$

Capacity of the 3D Deepo Theater is $N = 250$ seats. The set of several following constraints is customized for this number. We calculate numerically an equilibrium arrival time as a function of λ_a by the method of Glazer and Hassin. The curve fitting shows that this function can be approximated as $t_0^{5/4} + (\frac{\lambda_a - 220}{30})^2 = 1$, $0 < t_0 < 1$ for $220 < \lambda_a < 250$ with taking $t_0 = 0$ for $\lambda_a \geq 250$ and $t_0 = 1$ for $\lambda_a \leq 220$. The equilibrium arrival time to the 3D show at $t^* + 2$ (denoted τ_{t^*+2}) is important because this arrival process may have interference with the departure process from the Dolphin Show, starting at time $t_{dep} = t^* + t_{tales}$. The duration of the Dolphin Show t_{tales} is 30 minutes. So, $t_{dep} = t^* + 1$. The adjusted arrival rate λ_{trun} is used for consistency with the approximated function. Additional binary variables regulate the range between 220 and 250. Thus, if $\tilde{\tilde{\lambda}}_{t^*+2} \geq 220$ and $\tilde{\tilde{\lambda}}_{t^*+2} \leq 250$, then $y_{1tr} = 1$; if $\tilde{\tilde{\lambda}}_{t^*+2} \leq 250$, then $y_{2tr} = 0$, and other combinations are satisfied in this block of the constraints:

$900\tau_{t^*+2}^{\frac{5}{4}} + \lambda_{trun}^2 - 440\lambda_{trun} + 47500 = 0$

$\lambda_{trun} = 220 + y_{1tr}\tilde{\tilde{\lambda}}_{t^*+2} - 220y_{1tr} + 30y_{2tr}$

$\tilde{\tilde{\lambda}}_{t^*+2} - 250y_{2tr} \leq 250$

$\tilde{\tilde{\lambda}}_{t^*+2} - 251 y_{2tr} \geq 0$

$\tilde{\tilde{\lambda}}_{t^*+2} - 220 y_{1tr} \geq 0$

$\tilde{\tilde{\lambda}}_{t^*+2} - 280\delta_{\lambda 220} \leq 220$

$\tilde{\tilde{\lambda}}_{t^*+2} + 251\delta_{\lambda 250} \geq 251$

$-\delta_{\lambda 220} + y_{1tr} \leq 0$

$-\delta_{\lambda 250} + y_{1tr} \leq 0$

$\delta_{\lambda 220} + \delta_{\lambda 250} - y_{1tr} \leq 1$

$y_{1tr} + y_{2tr} \leq 1$

Extra control near the 3D Theater is established ($\gamma_{3D_{t+1}} = 1$) in the case if two consequent "overloaded" shows are expected (i.e. if $\tilde{\tilde{\lambda}}_t > 250$ and $\tilde{\tilde{\lambda}}_{t+1} > 250$).

$\tilde{\tilde{\lambda}}_t - 250\delta_{\omega_t} \leq 250$, $t = 1...T_{3D} - 1$

$\tilde{\tilde{\lambda}}_{t+1} - 250\delta_{\nu_{t+1}} \leq 250$, $t = 1...T_{3D} - 1$

$-\delta_{\omega_t} + \gamma_{3D_{t+1}} \leq 0$, $t = 1...T_{3D} - 1$

$-\delta_{\nu_{t+1}} + \gamma_{3D_{t+1}} \leq 0$, $t = 1...T_{3D} - 1$

$\delta_{\omega_t} + \delta_{\nu_{t+1}} - \gamma_{3D_{t+1}} \leq 1$, $t = 1...T_{3D} - 1$

The departure process from the Dolphin Theater starts at t_{dep}. During T_d=15 minutes $\tilde{N}$ visitors are required to leave the theater. This time T_d is discretized for periods ending at points $j = 1...T_{dep}$. We assume that blocking of corridor 1 cannot happen at $j = 1...T_{noblock} = \lceil \frac{C^{(1)}}{\Lambda} \rceil$, i.e. $\sum_{j=1}^{T_{noblock}} BP_j^{(1)} = 0$; and anti-blocking control ($\gamma_{dep_j} = 1$) can be organized after that. Otherwise, $BP_j^{(1)} \geq p_{C^{(1)}}$, where $p_{C^{(1)}}$ is obtained from the stationary model. The departure rate λ_{ex_j} remains at nonzero level Λ at least during $j = 1...T_\Lambda = \lceil \frac{\tilde{N}}{\Lambda} \rceil$ with the requirement $\sum_{j=1}^{T_{dep}} \lambda_{ex_j}(1 - BP_j^{(1)}) \geq \tilde{N}$. The special exit can be open starting at t_{dep} until $T_{ex} \leq T_d$, and considered as open with the same time discretization at $j = 1...T_{spec} \leq T_{dep}$. The blocking probabilities in corridor 2 after applying the stationary model can be approximated as $\frac{30((1-\alpha_j)-0.858)}{\sqrt{3600((1-\alpha_j)-0.858)^2+1}}+0.5 \leq BP_j^{(2)} \leq 1$. It is convenient to denote $3600(0.142-\alpha_j)^2$ as $\tilde{\alpha}_j$. So,

$l_{j+1}^{(1)} = l_j^{(1)} + \lambda_{ex_j}(1 - BP_j^{(1)}) - v_j^{(1)},\ j = 1...T_{dep} - 1$

$l_1^{(1)} = \Lambda$

$v_j^{(1)} \leq \frac{V_1}{L^{(1)}} l_j^{(1)} + \frac{V_1}{C^{(1)}L^{(1)}} l_j^{(1)} - \frac{V_1}{C^{(1)}L^{(1)}} (l_j^{(1)})^2,\ j = 1...T_{dep}$

$\sum_{j=1}^{T_{noblock}} BP_j^{(1)} = 0$

$p_{C^{(1)}}(1 - \gamma_{dep_j}) \leq BP_j^{(1)},\ j = T_{noblock} + 1...T_{dep}$

$\sum_{j=1}^{T_{noblock}} \gamma_{dep_j} = 0$

$\sum_{j=1}^{T_{dep}} \lambda_{ex_j}(1 - BP_j^{(1)}) \geq \tilde{N}$

$l_{j+1}^{(2)} = l_j^{(2)} + v_j^{(1)} - v_j^{(1)} BP_j^{(2)} - \alpha_j v_j^{(1)} + \alpha_j v_j^{(1)} BP_j^{(2)} - v_j^{(2)},\ j = 1...T_{dep} - 1$

$l_1^{(2)} = v_1^{(1)}(1 - BP_1^{(2)})(1 - \alpha_1)$

$v_j^{(2)} \leq \frac{V_1}{L^{(2)}} l_j^{(2)} + \frac{V_1}{C^{(2)}L^{(2)}} l_j^{(2)} - \frac{V_1}{C^{(2)}L^{(2)}} (l_j^{(2)})^2,\ j = 1...T_{dep}$

$l_j^{(1)} + l_j^{(2)} \leq l_{max},\ j = 1...T_{dep}$

$\tilde{\alpha}_j^2 (BP_j^{(2)})^2 - 4.26\alpha_j^{0.5} + 30\alpha_j \tilde{\alpha}_j^{0.5} - 0.5\tilde{\alpha}_j + 0.25 \geq 0,\ j = 1...T_{dep}$

$\tilde{\alpha}_j = 3600\alpha_j^2 - 1022.4\alpha_j + 73.5904,\ j = 1...T_{dep}$

$BP_j^{(1)} \leq 1,\ j = 1...T_{dep}$

$BP_j^{(2)} \leq 1,\ j = 1...T_{dep}$

$\alpha_j \leq \hat{\alpha}_j,\ j = 1...T_{dep}$

The last block of constraints reflects necessity to avoid or reduce collision the departure stream from the Dolphin Theater via the special exit with visitors of the 3D show. If we expect the full 3D theater ($\tilde{\tilde{\lambda}}_{t^*+1} > 250$), then the special exit is closed earlier ($\sum_{j=\lfloor \frac{2}{3} T_{dep} \rfloor + 1}^{T_{dep}} \alpha_j = 0$). Moreover, if the 3D show at $t^* + 2$ is also expected to be popular and $\tau_{t^*+2} \leq \tau_{cr}$ with a critical parameter $\tau_{cr} \leq 0.8$, then the special exit has extra restrictions ($\sum_{j=\lfloor \frac{1}{2} T_{dep} \rfloor + 1}^{T_{dep}} \alpha_j = 0$).

$\sum_{j=\lfloor \frac{2}{3} T_{dep} \rfloor + 1}^{T_{dep}} \alpha_j + 0.001\delta_\alpha \geq 0.001$

$\sum_{j=\lfloor \frac{2}{3} T_{dep} \rfloor + 1}^{T_{dep}} \alpha_j \leq (T_{dep} - \lfloor \frac{2}{3} T_{dep} \rfloor)(1 - \delta_\alpha)$

$\delta_{\omega_{t^*+1}} \leq \delta_\alpha$

$\sum_{j=\lfloor \frac{1}{2} T_{dep} \rfloor + 1}^{T_{dep}} \alpha_j + 0.001\delta_{\alpha\alpha} \geq 0.001$

$\sum_{j=\lfloor \frac{1}{2} T_{dep} \rfloor + 1}^{T_{dep}} \alpha_j \leq (T_{dep} - \lfloor \frac{1}{2} T_{dep} \rfloor)(1 - \delta_{\alpha\alpha})$

$\tau_{t^*+2} + (\tau_{cr} + 0.001)\delta_\tau \geq \tau_{cr} + 0.001$

$-\delta_{\omega_{t^*+1}} + \delta_{\alpha\alpha} \leq 0$

$-\delta_\tau + \delta_{\alpha\alpha} \leq 0$

$\delta_{\omega_{t^*+1}} + \delta_\tau - \delta_{\alpha\alpha} \leq 1$

The objective function is to minimize the work time of all servers/workers with weighted coefficients (”costs”):

min $k_1 \sum_{i=1}^{T} c_i + k_2 \sum_{i=1}^{T} \gamma_{esc_i} + k_3 \sum_{i=1}^{T} \gamma_i + k_4 \sum_{t=1}^{T_{3D}-1} \gamma_{3D_{t+1}} + k_5 \sum_{j=1}^{T_{dep}} \gamma_{dep_j} + k_6 \sum_{j=1}^{T_{dep}} \hat{\alpha}_j$.

4.6 Computational Experience and Managerial Insights

This section describes practical use of our model. As expected, demand for scheduled shows determines necessity of queueing control. Indeed, low demand for the Dolphin Show at t^* and 3D shows at and around that time guarantees the decomposed variant, i.e. three stochastic submodels work independently or almost independently. If this is not the case, the decisions are interconnected and can be trade-offs.

The developed large scale MINLP in the form of Signomial Programming problem with mixed variables contains numerous parameters, and its consistency with the previous study is the primary goal in providing the input characteristics. The parameters of time discretization T and T_{dep} reflect the trade-off between the size of the problem and approximation quality of stochastic submodels. For example, one natural choice is $T = 60$ (i.e. one hour before t^* has 1-minute intervals) and $T_{dep} = 30$ (i.e. 15-minute period after the same Dolphin Show have 30-second intervals) creates MINLP with about 1,000 constraints and about 1,000 variables (and a half of them are binary). Besides, such an instance contains approximately 1,000 positive and 1,000 negative signomial terms with about 500 participating variables in each case.

It is a big challenge for modern software to solve such instances. Knowing the underlying physical processes, we are able to construct feasible solutions as the upper bounds of optimal solutions with accuracy of 10^{-13} for nonlinear equality constraints using, for example, Microsoft Excel. Convexification techniques from the previous two chapters provide the lower bounds.

The major concern following from our analysis is the prescription for visitors to arrive 30 minutes prior to the scheduled show time at the Dolphin Theater (see Figure 15). It creates the nonlinear peak of arrivals. Spectators do not have assigned seats, and what is printed on the tickets triggers their behavior towards congestions. A possible remedy is to replace "30 minutes" with a few different numbers, say 35 minutes for first 600 ticket buyers, 30 minutes for next 600, and 25 minutes for the remainder of people. Clearly, it is impossible to apply a precise minute-by-minute appointment system, but imitations of it can mitigate accumulation of people in the main lobby. These numbers in the suggested approach can be adjusted to the expected arrivals to 3D shows.

Another managerial insight is related to steps towards "uniformity" of demand for 3D shows. In particular, it is necessary to avoid the situations of mass postponing of the decision to visit a 3D show simultaneously with natural growing of this demand in some periods (i.e. even without taking into account the Dolphin Tales). The demand for 3D shows in the middle of the day is always higher than that in the morning and late afternoon times. Visitors do not spend the whole day in the aquarium, and their desire to attend a 3D show approximately correlates with general arrival statistics to the whole aquarium. Decisions to control queueing to the Dolphin Tales can help with avoiding increasing the peaks of arrivals to 3D shows.

The constructed Signomial Programming problem gives a wide avenue for future research in computational aspects of developing and applying techniques after convexification. Varying parameters may give a ground in further investigation of approximation approaches. Proposed and modified stochastic submodels can be verified and have extended development in other applications, especially in crowd control and emergency preparedness, and healthcare operations.

CHAPTER V

CONCLUSION

This thesis provides a new theoretical framework for obtaining facet-defining inequalities from conflict graphs and hypergraphs. We introduce a mixed hyperedge method. It represents mixed conflict hypergraphs - a new research area, which combines theories of mixed conflict graphs and 0-1 conflict hypergraphs. Besides, the novel names mixed star-clique, weighted complementary, and mixed incomplete linking inequalities appear in this work.

The mixed hyperedge method is successfully developed in consideration of two structured problems. We outperform modern commercial MIP software in the branch-and-cut framework. Those programs are building blocks in solving special Mixed Integer Nonlinear Programming problems.

A practical application of signomial programming arises in our study of managing guest flows in Georgia Aquarium. We also demonstrate how different tools of Operations Research like optimization, stochastic processes, and queueing theory are combined in modeling real-world events and processes.

Next research steps can be in all mentioned topics. Conflict graphs and hypergraphs are fundamental and have slow development with the time. The introduced mixed hyperedge method has been applied to two structured problems so far and needs extra research in terms of applicability to other problems, attempting for abstract generalizations, possibility to be used directly in some nonlinear problems. Convexification of special nonlinear problems with discrete variables is only one of the building blocks towards solving those problems, and it is valuable to conduct research in connections and development of other parts of the solution process. At last, nonlinear modeling is widely used in general and the scope of the considered applied areas can be extended.

APPENDIX A

DERIVING INEQUALITIES OF SET S IN CHAPTER II

Here we provide the details of refining and adjusting the initial inequalities with "big-M's" and "small m's" to constraints (10)-(18) of set S (initially constraints (1)-(9)). Our approach offers 4 variants for constraints (13), (14), (16), and (17) in the form of $\bar{\tilde{q}} + binary$, $\tilde{q} + binary$, $\bar{\tilde{q}} + \overline{binary}$, and $\tilde{q} + \overline{binary}$. Inequalities marked as (✓) have the structure suitable for MVP (positive coefficients, addition only, $\leq$, etc.).

- Constraint (10) $y_{ij} \leq t_j \quad \forall i, j$ is also $y_{ij} + \bar{t}_j \leq 1$ (✓)

- Constraint (11) $\sum_{j=1}^{n} s_{ij} \leq 1 \quad \forall i$ is $\sum_{j \in J_i^+} s_{ij} \leq 1$ (✓)

- Constraint (12) $\sum_{j=1}^{n} a_{ij} q_{ij} + \tilde{m}_i \sum_{j=1}^{n} s_{ij} \geq \tilde{m}_i + 1 \quad \forall i$

  $\Longrightarrow \sum_{j \in J_i^+} a_{ij} q_{ij} + \sum_{j \in J_i^-} a_{ij} + \tilde{m}_i \sum_{j \in J_i^+} s_{ij} \geq \tilde{m}_i + 1 \Longrightarrow \tilde{m}_i + 1 = -u \sum_{j \in J_i^+} a_{ij} + \sum_{j \in J_i^-} a_{ij}$

  $\Longrightarrow \sum_{j \in J_i^+} a_{ij} q_{ij} + u \sum_{j \in J_i^+} a_{ij} + (\sum_{j \in J_i^-} a_{ij} - u \sum_{j \in J_i^+} a_{ij} - 1) \sum_{j \in J_i^+} s_{ij} \geq 0.$

  So, $-\sum_{j \in J_i^+} a_{ij} \tilde{q}_{ij} + (-\sum_{j \in J_i^-} a_{ij} + u \sum_{j \in J_i^+} a_{ij} + 1) \sum_{j \in J_i^+} s_{ij} \leq 0.$

  or $\sum_{j \in J_i^+} a_{ij} \bar{\tilde{q}}_{ij} + (-\sum_{j \in J_i^-} a_{ij} + u \sum_{j \in J_i^+} a_{ij} + 1) \sum_{j \in J_i^+} s_{ij} \leq 2u \sum_{j \in J_i^+} a_{ij}$ (✓).

  We assume here that $-\sum_{j \in J_i^-} a_{ij} + u \sum_{j \in J_i^+} a_{ij} + 1 \leq 2u \sum_{j \in J_i^+} a_{ij}$, i.e. $u \sum_{j \in J_i^+} a_{ij} \geq 1 - \sum_{j \in J_i^-} a_{ij}$. Otherwise, power p does not exist for term i ($\sum_{j \in J_i^+} s_{ij} = 0$) and we have to accept only negative transformations. Thus, we can see an alternative way in choosing u depending on a_{ij}: $u_i = \frac{1 - \sum_{j \in J_i^-} a_{ij}}{\sum_{j \in J_i^+} a_{ij}}$ for each posynomial term i or for all i $\hat{u} = \max_i (\frac{1 - \sum_{j \in J_i^-} a_{ij}}{\sum_{j \in J_i^+} a_{ij}})$ with the requirement of $u > 1$.

  Thus, for practical purposes we can choose $u = \ulcorner max(u^*, \hat{u}) \urcorner$ or admit for some terms negative transformations only to avoid u becoming large.

- Constraint (13) $q_{ij} + m_1 s_{ij} \geq m_1 + 1 \;\; \forall i, j \; s.t. \; a_{ij} > 0$

  $m_1 = -u - 1 \implies q_{ij} \geq (u+1)s_{ij} - u \; \forall i, j \in J_i^+$

  So, $\bar{\tilde{q}}_{ij} + (u+1)s_{ij} \leq 2u$ ($\checkmark$)

  or $-\tilde{q}_{ij} + (u+1)s_{ij} \leq 0$

  or $\bar{\tilde{q}}_{ij} + (u+1)(1 - \bar{s}_{ij}) \leq 2u$

  or $-\tilde{q}_{ij} + (u+1)(1 - \bar{s}_{ij}) \leq 0$

- Constraint (14) $q_{ij} \leq \varepsilon(s_{ij} - 1) - m_2 s_{ij} \;\; \forall i, j \; s.t. \; a_{ij} > 0$

  $m_2 = -u \implies q_{ij} \leq \frac{1}{u}(s_{ij} - 1) + u s_{ij} \implies q_{ij} \leq (u + \frac{1}{u})s_{ij} - \frac{1}{u} \; \forall i, j \in J_i^+$

  So, $-\bar{\tilde{q}}_{ij} - (u + \frac{1}{u})s_{ij} \leq -(u + \frac{1}{u})$

  or $\tilde{q}_{ij} - (u + \frac{1}{u})s_{ij} \leq u - \frac{1}{u}$

  or $-\bar{\tilde{q}}_{ij} + (u + \frac{1}{u})\bar{s}_{ij} \leq 0$

  or $\tilde{q}_{ij} + (u + \frac{1}{u})\bar{s}_{ij} \leq 2u$ ($\checkmark$)

- Constraint (15) $y_{ij} + s_{ij} \geq 1 \;\; \forall i, j \; s.t. \; a_{ij} > 0$ is also $\bar{y}_{ij} + \bar{s}_{ij} \leq 1 \; \forall i, j \in J_i^+$

- Constraint (16) $q_{ij} - 1 \geq m_1 y_{ij} \;\; \forall i, j \; s.t. \; a_{ij} > 0$

  $m_1 = -u - 1 \implies q_{ij} + (u+1)y_{ij} \geq 1 \; \forall i, j \in J_i^+$

  So, $\bar{\tilde{q}}_{ij} - (u+1)y_{ij} \leq u - 1$

  or $-\tilde{q}_{ij} - (u+1)y_{ij} \leq -(u+1)$

  or $\bar{\tilde{q}}_{ij} + (u+1)\bar{y}_{ij} \leq 2u$ ($\checkmark$)

  or $-\tilde{q}_{ij} + (u+1)\bar{y}_{ij} \leq 0$

- Constraint (17) $q_{ij} - 1 \leq M_1 y_{ij} \;\; \forall i, j \; s.t. \; a_{ij} > 0$

  $M_1 = u - 1 \implies q_{ij} \leq (u-1)y_{ij} + 1 \; \forall i, j \in J_i^+$

  So, $-\bar{\tilde{q}}_{ij} - (u-1)y_{ij} \leq -(u-1)$

  or $\tilde{q}_{ij} - (u-1)y_{ij} \leq u + 1$

  or $-\bar{\tilde{q}}_{ij} + (u-1)\bar{y}_{ij} \leq 0$

  or $\tilde{q}_{ij} + (u-1)\bar{y}_{ij} \leq 2u$ ($\checkmark$)

- Constraint (18) $y_{ij} \leq (1-\varepsilon)q_{ij} + M(1-s_{ij}) \;\; \forall i, j \; s.t. \; a_{ij} > 0$

  $\Longrightarrow y_{ij} \leq (1-\frac{1}{u})q_{ij} + u(1-s_{ij}) \Longrightarrow \frac{1-u}{u}q_{ij} + y_{ij} + us_{ij} \leq u \; \forall i, j \in J_i^+$

  So, $\frac{u-1}{u}\bar{\tilde{q}}_{ij} + y_{ij} + us_{ij} \leq 2u - 1 \Longrightarrow \bar{\tilde{q}}_{ij} + \frac{u}{u-1}y_{ij} + \frac{u^2}{u-1}s_{ij} \leq 2u + \frac{u}{u-1}$ (✓)

  or $-\tilde{q}_{ij} + \frac{u}{u-1}y_{ij} + \frac{u^2}{u-1}s_{ij} \leq \frac{u}{u-1}$

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